Online Appendix for *Dynamic Information Aggregation: Learning from the Past*

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Contents

A. Proofs in the Main Text

A.1 Proof of Simple Lemmas

A.1.1 Proof of Lemma 2.1

Recall the two equations that define the beauty-contest game,

$$p_{it} = (1-\alpha)\mathbb{E}_{it}[q_t] + \alpha\mathbb{E}_{it}[p_t], \quad \text{and} \quad p_t = \int p_{it}\,\mathrm{d}i.$$

Using

$$\overline{\mathbb{E}}_t^0[q_t] \equiv q_t, \quad \text{and} \quad \overline{\mathbb{E}}_t^1[q_t] \equiv \int \mathbb{E}_{it}\left[\overline{\mathbb{E}}_t^0[q_t]\right]\mathrm{d}i,$$

it follows that

$$p_t = \overline{\mathbb{E}}_t^1[q_t] + \alpha \int \mathbb{E}_{it}[p_t].$$

Next, substitute p_t into itself to get

$$p_t = \overline{\mathbb{E}}_t^1[q_t] + \alpha \int \mathbb{E}_{it}\left[\overline{\mathbb{E}}_t^1[q_t]\right] + \alpha^2 \int \mathbb{E}_{it}\left[\int \mathbb{E}_{it}[p_t]\right].$$

Iterating on this last step and using

$$\overline{\mathbb{E}}_t^{k+1}[q_t] \equiv \int \mathbb{E}_{it}\left[\overline{\mathbb{E}}_t^k[q_t]\right]\mathrm{d}i,$$

yields the result.

A.1.2 Proof of Lemma 3.1

For a discussion of invertibility and its connection to the inside roots of a stochastic ARMA process see Brockwell and Davis (2002), Section 3.1. When the price process is invertible, observing the history of prices, $p^t \equiv \{p_{t-k}\}_{k=0}^{\infty}$ reveals the history of shocks to the fundamental, that is $p_{t-k} = g(L)\eta_{t-k}$ implies $\eta_{t-k} = g(L)^{-1}p_{t-k}$, for all $k \geq 0$. Since $q_{t-k} = \rho q_{t-k-1} + \eta_{t-k}$, it follows that $\eta_{t-k} = q_{t-k}/(1-\rho L)$, which implies the result.

A.2 Proof of Proposition 3.1

Suppose that the stochastic process for p_t is invertible, then observing $\{p_k\}_{k=-\infty}^{t-1}$ perfectly reveals the underlying aggregate shocks $\{\eta_k\}_{k=-\infty}^{t-1}$ and, therefore, $\{q_k\}_{k=-\infty}^{t-1}$, so that the only shock the firms are uncertain about is the current η_t. It follows that the information structure is exogenous and that the invertible equilibrium is unique. Guess that the equilibrium policy function has state variables x_{it} and q_{t-1}, that is guess that firm i's policy function can be written as

$$p_{it} = \phi_x x_{it} + \phi_q q_{t-1},$$

for some scalars ϕ_x and ϕ_q. It follows that, in aggregate terms,

$$p_t = \phi_x q_t + \phi_q q_{t-1}.$$

To verify the guess, notice that, since $x_{it} - \rho q_{t-1}$ is a noisy signal about η_t with precision τ, we have that

$$\mathbb{E}_{it}[q_t] = \frac{\rho}{1+\tau} q_{t-1} + \frac{\tau}{1+\tau} x_{it}.$$

Substituting these results into the best-response function (2) we obtain

$$p_{it} = \frac{\tau((1-\alpha)+\alpha\phi_x)}{1+\tau} x_{it} + \left(\frac{\rho((1-\alpha)+\alpha\phi_x)}{1+\tau} + \alpha\phi_q \right) q_{t-1},$$

which implies the following consistency requirement

$$\phi_q = \frac{\rho}{1+(1-\alpha)\tau}, \qquad \text{and} \qquad \phi_x = \frac{(1-\alpha)\tau}{1+(1-\alpha)\tau}.$$

Hence, for p_t to indeed follow an invertible process it is necessary and sufficient that $g(L) = \frac{\phi_x + \phi_q L}{1-\rho L}$ not have an inside root, or that $|\phi_q/\phi_x| < 1$ which implies the result.

A.3 Proof of Proposition 3.2

The first claim follows directly from the proof of Proposition 3.1. To establish the second claim we follow a significantly more involved argument. To facilitate reading it we include the proof of the necessary lemmas at the end of this section.

For a contradiction, suppose there is a finite-state representation, then the law of motion of the aggregate action can be written as

$$p_t = g(L)\eta_t = C(L)h(L)\eta_t,$$

where $h(L)$ is analytic and does not contain any inside root, and $C(z)$ is given by

$$C(z) = \prod_{i=1}^{n} (z - \lambda_i).$$

where $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ are inside roots of $g(z)$. The signal structure can be written as follows

$$\begin{bmatrix} x_{it} \\ p_{t-1} \end{bmatrix} = \mathbf{\Gamma}(L) \begin{bmatrix} u_{it} \\ \eta_t \end{bmatrix} \equiv \begin{bmatrix} \tau^{-1/2} & \frac{1}{1-\rho L} \\ 0 & Lg(L) \end{bmatrix} \begin{bmatrix} u_{it} \\ \eta_t \end{bmatrix}.$$

The determinant of $\mathbf{\Gamma}(L)$ is

$$\det[\mathbf{\Gamma}(L)] = \tau^{-1/2} L C(L) h(L),$$

and it contains inside roots $\{\lambda_1, \ldots, \lambda_n, \lambda_{n+1}\}$, with $\lambda_{n+1} \equiv 0$. Denote the Blaschke matrix by

$$\mathbf{B}(L;\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1-\lambda L}{L-\lambda} \end{bmatrix},$$

and let the fundamental representation of the signal process be given by

$$\mathbf{\Gamma}^{*}(L)\boldsymbol{\epsilon}_{it} = \mathbf{\Gamma}(L)\begin{bmatrix} u_{it} \\ \eta_t \end{bmatrix},$$

where

$$\boldsymbol{\epsilon}_{it} \equiv \mathbf{A}(L)\begin{bmatrix} u_{it} \\ \eta_t \end{bmatrix}, \quad \mathbf{A}(L) \equiv \mathbf{B}'\left(L^{-1};\lambda_{n+1}\right)\mathbf{W}'_{\lambda_{n+1}}\cdots\mathbf{B}'\left(L^{-1};\lambda_1\right)\mathbf{W}'_{\lambda_1}, \quad \mathbf{\Gamma}^{*}(L) \equiv \mathbf{\Gamma}\left(L\right)\mathbf{A}'(L^{-1}).$$

and $\{\mathbf{W}_{\lambda_i}\}$ are the rotation matrices that satisfy $\mathbf{W}_{\lambda_i}\mathbf{W}'_{\lambda_i} = \mathbf{I}$. Next, define the following matrices recursively

$$\begin{aligned} \mathbf{\Gamma}_0(L) &\equiv \mathbf{\Gamma}(L), \\ \mathbf{\Gamma}_k(L) &\equiv \mathbf{\Gamma}_{k-1}(L)\mathbf{W}_{\lambda_k}\mathbf{B}(L;\lambda_k). \end{aligned}$$

The following lemma characterizes useful properties of $\mathbf{\Gamma}_k(L)$.

Lemma A.1. *The matrix $\mathbf{\Gamma}_k(L)$ is given by*

$$\mathbf{\Gamma}_k\left(L\right) = \begin{bmatrix} \gamma_1^k\left(L\right) & \gamma_2^k\left(L\right) \\ \gamma_3^k\left(L\right)g\left(L\right) & \gamma_4^k\left(z\right)g\left(L\right) \end{bmatrix}$$

with all $\gamma_i^k(L)$ independent of $g(L)$. Moreover,

$$\delta_k\left(z\right) \equiv \frac{\gamma_1^k\left(z\right)}{\gamma_2^k\left(z\right)}$$

satisfies the following recursive structure:

$$\delta_0(z) = \tau^{-1/2}(1-\rho z), \tag{A.1}$$

$$\delta_k\left(z\right) = \frac{1+\delta_{k-1}\left(\lambda_k\right)\delta_{k-1}\left(z\right)}{\delta_{k-1}\left(\lambda_k\right)-\delta_{k-1}\left(z\right)}\frac{z-\lambda_k}{1-\lambda_k z}. \tag{A.2}$$

Finally, for $k \geq 2$, there exists some constant d_k such that

$$\delta_{k-1}\left(z\right) = \delta_{k-1}\left(\lambda_k\right) + d_k\frac{z-\lambda_k}{1-\lambda_{k-1}z}. \tag{A.3}$$

Using the recursive structure of $\delta_k(z)$, it is straightforward to verify that $\mathbf{A}(z)$ can be written as

$$\mathbf{A}\left(z\right) = c_1\mathbf{\Phi}(z) = \mathbf{A}_{n+1}\left(z\right)\mathbf{A}_n\left(z\right)\ldots\mathbf{A}_2\left(z\right)\mathbf{A}_1\left(z\right),$$

with

$$\begin{aligned} c_1 &= \prod_{k=1}^{n+1}\frac{\gamma_2^{k-1}\left(\lambda_k\right)}{\sqrt{\gamma_1^{k-1}\left(\lambda_k\right)^2+\gamma_2^{k-1}\left(\lambda_k\right)^2}}, \\ \mathbf{A}_k\left(z\right) &= \frac{\sqrt{\gamma_1^{k-1}\left(\lambda_k\right)^2+\gamma_2^{k-1}\left(\lambda_k\right)^2}}{\gamma_2^{k-1}\left(\lambda_k\right)}\mathbf{B}'\left(z^{-1};\lambda_k\right)\mathbf{W}'_{\lambda_k} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{z-\lambda_k}{1-\lambda_k z} \end{bmatrix}\begin{bmatrix} \delta_{k-1}\left(\lambda_k\right) & 1 \\ -1 & \delta_{k-1}\left(\lambda_k\right) \end{bmatrix}. \end{aligned}$$

The following lemma characterizes a useful property of $\mathbf{\Phi}(z)$.

Lemma A.2. *The elements of $\mathbf{\Phi}(z)$ satisfy*

$$\Phi_{12}(z) = -\frac{\prod_{k=1}^{n+1}\left(\delta_{k-1}\left(\lambda_k\right)^2+1\right)}{c_2}\frac{\delta_0(z^{-1})}{H(z^{-1})} + \Phi_{22}(z)\delta_{n+1}(z^{-1}),$$
$$\Phi_{22}(z) = c_2\frac{H(z)\delta_0(z^{-1})\delta_{n+1}(z)+1}{\delta_0(z)\delta_0(z^{-1})+1},$$

where $H(z) \equiv \prod_{k=1}^{n+1}\frac{z-\lambda_k}{1-\lambda_k z}$, and c_2 is some constant.

The equivalence result in Huo and Pedroni (2020) implies that

$$p_{it} = \widetilde{\mathbb{E}}_{it}[q_t],$$

where $\widetilde{\mathbb{E}}_{it}$ is the expectation conditional on the same information set but with a precision of private signals misperceived to be $\tau \equiv (1-\alpha)\tau$. Then, the aggregate action is

$$p_t = \int \widetilde{\mathbb{E}}_{it}[q_t]\,\mathrm{d}i.$$

Since the Hansen-Sargent formula implies

$$\widetilde{\mathbb{E}}_{it}\left[q_t\right] = \frac{1}{\rho}\left(\frac{\mathbf{\Gamma}\left(L\right)}{L} - \frac{\mathbf{\Gamma}^*\left(0\right)\mathbf{A}\left(L\right)}{L}\right)_{1^{st}\text{ row}}\begin{bmatrix} u_{it} \\ \eta_t \end{bmatrix},$$

we obtain the following fixed point problem,

$$g(z) = \frac{1}{\rho}\left[\frac{\mathbf{\Gamma}\left(z\right)}{z} - \frac{\mathbf{\Gamma}^*\left(0\right)\mathbf{A}\left(z\right)}{z}\right]_{12} = \frac{1}{\rho z}\left(\frac{1}{1-\rho z} - S(z)\right),$$

or

$$h(z) = \frac{1}{\rho}\frac{1}{zC(z)}\left(\frac{1}{(1-\rho z)} - S(z)\right),$$

where

$$S(z) \equiv \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{\Gamma}^*\left(0\right)\mathbf{A}\left(z\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1\Big(\gamma_1^{n+1}(0)\Phi_{12}(z) + \gamma_2^{n+1}(0)\Phi_{22}(z)\Big).$$

For $h(z)$ to be an equilibrium, it has to be that

$$\frac{1}{(1-\rho\lambda_i)} = S(\lambda_i), \quad \text{for all } i \in \{1,\ldots,n\}, \tag{A.4}$$

so that the poles in $C(z)$ can be removed.

Next, we show that with a finite number of inside roots, there does not exist an equilibrium.

Case 1: $n = 1$.. First let $x \equiv \rho + \frac{1+\tau}{\rho}$ and notice that $|\rho| < 1$ and $\tau > 0$ imply $|x| > 2$. In this case $S(z)$ takes a simple form and we can calculate

$$S(\lambda_1) - \frac{1}{1-\rho\lambda_1} = \frac{\tau\lambda_1}{(1-\lambda_1\rho)} \frac{x-\lambda_1}{\left(1-\lambda_1^2\right) + (x-2\lambda_1)(\rho - x)},$$

so that equation (A.4) implies that $\lambda_1 = x$ which is outside the unit circle.

Case 2: $n = 2$.. Suppose that $\lambda_1 \neq \lambda_2$, then

$$\frac{S(\lambda_1)}{S(\lambda_2)} - \frac{1-\rho\lambda_2}{1-\rho\lambda_1} = \frac{\tau(\lambda_2 - \lambda_1)}{(1-\rho\lambda_1)} \frac{(x-\lambda_1)(x-\lambda_2) - 1}{\left(1-\lambda_1^2\right)(\rho - x + \lambda_2) - (x-2\lambda_1)((x-\lambda_2)(\rho - x) + 1)},$$

and equation (A.4) for $i = \{1, 2\}$ implies that $(x-\lambda_1)(x-\lambda_2) = 1$, which implies that either $|\lambda_1| > 1$ or $|\lambda_2| > 1$. Next, if $\lambda_1 = \lambda_2 = \lambda$, we have that

$$S(\lambda) - \frac{1}{1-\rho\lambda} = \frac{\tau\lambda}{1-\lambda\rho} \frac{-3\lambda^2 + 3x\lambda - \left(x^2 - 1\right)}{(1-\lambda^2)(4\lambda - x(1+\lambda^2)) + (4x\lambda - x^2 + \lambda^4 - 6\lambda^2 + 1)(\rho - x)},$$

and equation (A.4) implies that $3\lambda^2 - 3x\lambda + x^2 = 1$. Notice that the discriminant of this quadratic equation on λ is $9x^2 - 12(x^2 - 1)$ and that it is negative whenever $|x| > 2$. Therefore, the solutions are complex. Complex λ's are allowed but necessitate a conjugate which is not possible in this case since we have assumed $\lambda_1 = \lambda_2$.

Case 3: $n > 2$.. From the definition of $S(z)$ and Lemma A.2, it follows that

$$S(\lambda_i) = c_1 c_2 \gamma_2^{n+1}(0) \frac{1 + \delta_{n+1}(0)\delta_{n+1}(\lambda_i^{-1})}{1 + \delta_0(\lambda_i)\delta_0(\lambda_i^{-1})}.$$

Equation (A.3) together with the fact that $\lambda_{n+1} = 0$ implies that

$$\delta_{n+1}(z) = \delta_{n+1}(0) + d_{n_1} z,$$

for some constant d_{n+1}. Thus, we can rewrite $S(\lambda_i)$ as

$$S(\lambda_i) = c_1 c_2 \gamma_2^{n+1}(0) \frac{1 + \delta_{n+1}(0)(\delta_{n+1}(0) + d_{n+1}\lambda_i^{-1})}{1 + \delta_0(\lambda_i)\delta_0(\lambda_i^{-1})}.$$

Suppose that the solution to the system of equations (A.4) includes λ_i, λ_j, λ_k different from each other and all inside the unit circle. It follows that

$$\frac{S(\lambda_i)}{S(\lambda_j)} = \frac{1-\rho\lambda_j}{1-\rho\lambda_i}, \quad \text{or} \quad \frac{1 + \delta_{n+1}(0)(\delta_{n+1}(0) + d_{n+1}\lambda_i^{-1})}{1 + \delta_0(\lambda_i)\delta_0(\lambda_i^{-1})} = \frac{1 + \delta_{n+1}(0)(\delta_{n+1}(0) + d_{n+1}\lambda_j^{-1})}{1 + \delta_0(\lambda_j)\delta_0(\lambda_j^{-1})},$$

which can be written as

$$\delta_{n+1}(0)\left(\tau + (1-\rho\lambda_i)\left(1-\rho\lambda_j\right)\right) + \left((1-\rho\lambda_i)\left(1-\rho\lambda_j\right) + \lambda_i\lambda_j\tau\right)\rho\left(1 + \delta_{n+1}^2(0)\right) = 0.$$

Suppose that $\tau + (1-\rho\lambda_i)\left(1-\rho\lambda_j\right) \neq 0$, then

$$\frac{\left(1-\lambda_i\lambda_j\right)\tau}{\tau + (1-\rho\lambda_i)\left(1-\rho\lambda_j\right)} = 1 - \frac{d_{n+1}\delta_{n+1}(0)}{\rho\left(1+\delta_{n+1}^2(0)\right)}.$$

Similarly, if $\tau + (1-\rho\lambda_i)(1-\rho\lambda_k) \neq 0$, then

$$\frac{(1-\lambda_i\lambda_k)\tau}{\tau + (1-\rho\lambda_i)(1-\rho\lambda_k)} = 1 - \frac{d_{n+1}\delta_{n+1}(0)}{\rho\left(1+\delta_{n+1}^2(0)\right)}.$$

Combining the two conditions above, we have

$$\frac{\left(1-\lambda_i\lambda_j\right)}{\tau + (1-\rho\lambda_i)\left(1-\rho\lambda_j\right)} = \frac{(1-\lambda_i\lambda_k)}{\tau + (1-\rho\lambda_i)(1-\rho\lambda_k)},$$

which implies

$$\tau\lambda_i + (1-\lambda_i\rho)(1\lambda_i - \rho) = 0 \quad \Rightarrow \quad 1 + \delta_0(\lambda_i)\delta_0(\lambda_i^{-1}) = 0.$$

This cannot be the case since $S(\lambda_i)$ must be a finite number. Therefore, if there exists a solution, it has to be that

$$\tau + (1-\rho\lambda_i)\left(1-\rho\lambda_j\right) = 0.$$

This condition, however, implies that λ_i and λ_j cannot be simultaneously within the unit circle.

Next, suppose there exits a solution with $\lambda_i = \lambda$, for all $i \in \{1,\ldots,n\}$ with $|\lambda| < 1$. Let $\{\lambda_{1,k}\}_{k=0}^{\infty}, \ldots, \{\lambda_{n,k}\}_{k=0}^{\infty}$ be n sequences such that $\lim_{k\to\infty}\lambda_{i,k} = \lambda$ and $\lambda_{i,k} \neq \lambda_{j,k}$ for all $i,j \in \{1,\ldots,n\}$ and all $k \geq 0$. Define

$$\omega_k \equiv \sum_{i=1}^{n}\left|S(\lambda_{i,k}) - \frac{1}{1-\rho\lambda_{i,k}}\right|.$$

By continuity of $S(\cdot)$, ω_k approaches 0 as k goes to infinity, since $\lambda_{i,k}$ approaches to λ for all $i \in \{1,\ldots,n\}$. However, since $\lambda_{i,k} \neq \lambda_{j,k}$, as established above, only if some $\lambda_{i,k}$ are outside the unit circle, can ω_k approach 0. Since $|\lambda| < 1$, as k goes to infinity, all $|\lambda_{i,k}| < \delta$ for any $\delta < 1$, which implies that ω_k cannot be close to zero. This is a contradiction. The case where all λ_i equal to each other, except for one, can be dealt with in a similar way, which concludes the proof.

A.3.1 Proof of Lemma A.1

We prove this lemma by induction. For $k = 0$, this is clearly the case. For $k \geq 1$, suppose that

$$\mathbf{\Gamma}_k(z) = \begin{bmatrix} \gamma_1^k(z) & \gamma_2^k(z) \\ \gamma_3^k(z)\,g(z) & \gamma_4^k(z)\,g(z) \end{bmatrix}.$$

Then, it follows that

$$\mathbf{\Gamma}_{k+1}(z) = \mathbf{\Gamma}_{k-1}(z)\,\mathbf{W}_{\lambda_{k+1}}\mathbf{B}(z;\lambda_{k+1}),$$

with

$$\mathbf{W}_{\lambda_{k+1}} = \frac{1}{\sqrt{\gamma_1^k(\lambda_{k+1})^2 + \gamma_2^k(\lambda_{k+1})^2}} \begin{bmatrix} \gamma_1^k(\lambda_{k+1}) & -\gamma_2^k(\lambda_{k+1}) \\ \gamma_2^k(\lambda_{k+1}) & \gamma_1^k(\lambda_{k+1}) \end{bmatrix}.$$

So that

$$\begin{aligned}
\mathbf{\Gamma}_{k+1}(z) =& \frac{1}{\sqrt{\gamma_1^k(\lambda_{k+1})^2 + \gamma_2^k(\lambda_{k+1})^2}} \begin{bmatrix} \gamma_1^k(z) & \gamma_2^k(z) \\ \gamma_3^k(z)\,g(z) & \gamma_4^k(z)\,g(z) \end{bmatrix} \begin{bmatrix} \gamma_1^k(\lambda_{k+1}) & -\gamma_2^k(\lambda_{k+1}) \\ \gamma_2^k(\lambda_{k+1}) & \gamma_1^k(\lambda_{k+1}) \end{bmatrix} \\
=& \frac{\begin{bmatrix} \left(\gamma_1^k(\lambda_{k+1})\gamma_1^k(z) + \gamma_2^k(\lambda_{k+1})\gamma_2^k(z)\right) & \left(\gamma_1^k(\lambda_{k+1})\gamma_2^k(z) - \gamma_2^k(\lambda_{k+1})\gamma_1^k(z)\right)\frac{1-z\lambda_{k+1}}{z-\lambda_{k+1}} \\ \left(\gamma_1^k(\lambda_{k+1})\gamma_2^k(z) + \gamma_2^k(\lambda_{k+1})\gamma_4^k(z)\right)g(z) & \left(\gamma_1^k(\lambda_{k+1})\gamma_4^k(z) - \gamma_2^k(\lambda_{k+1})\gamma_3^k(z)\right)\frac{1-z\lambda_{k+1}}{z-\lambda_{k+1}}g(z) \end{bmatrix}}{\sqrt{\gamma_1^k(\lambda_{k+1})^2 + \gamma_2^k(\lambda_{k+1})^2}}
\end{aligned}$$

which has the desired structure. Note, moreover, that γ_1^{k+1} and γ_2^{k+1} satisfy

$$\begin{aligned}
\gamma_1^{k+1}(z) &= \frac{\gamma_1^k(\lambda_{k+1})\gamma_1^k(z) + \gamma_2^k(\lambda_{k+1})\gamma_2^k(z)}{\sqrt{\gamma_1^k(\lambda_{k+1})^2 + \gamma_2^k(\lambda_{k+1})^2}}, \\
\gamma_2^{k+1}(z) &= \frac{\gamma_1^k(\lambda_{k+1})\gamma_2^k(z) - \gamma_2^k(\lambda_{k+1})\gamma_1^k(z)}{\sqrt{\gamma_1^k(\lambda_{k+1})^2 + \gamma_2^k(\lambda_{k+1})^2}} \frac{1 - z\lambda_{k+1}}{z - \lambda_{k+1}}.
\end{aligned}$$

It follows from these recursions that $\delta_k(z)$ satisfies equation (A.2). To prove equation (A.3), first notice that, for $k = 1$, it follows from (A.1) and (A.2) that

$$\delta_1(z) = \delta_1(\lambda_1) + \frac{(1-\rho\lambda_1)(\lambda_1-\rho) + \lambda_1\tau}{\sqrt{\tau}\rho\left(1-\lambda_1^2\right)} \frac{z-\lambda_1}{1-\lambda_1 z}.$$

Next, suppose that there exists d_k such that

$$\delta_k(z) = \delta_k(\lambda_{k+1}) + d_k \frac{z - \lambda_{k+1}}{1 - \lambda_k z},$$

then, equation (A.2) implies that

$$\delta_{k+1}(z) = \delta_{k+1}(\lambda_{k+1}) + \frac{\left(1 + \delta_k(\lambda_{k+1})^2\right)(\lambda_k - \lambda_{k+1}) - d_k\left(1 - \lambda_{k+1}^2\right)\delta_k(\lambda_{k+1})}{d_k\left(1 - \lambda_{k+1}^2\right)} \frac{z - \lambda_{k+1}}{1 - \lambda_{k+1} z},$$

which, again by induction, establishes the result. □

A.3.2 Proof of Lemma A.2

From the definition of $\mathbf{A}_k(z)$ and equation (A.2), it follows that

$$\mathbf{A}_k(z) \begin{bmatrix} 1 \\ -\delta_{k-1}(z) \end{bmatrix} = \left(\delta_{k-1}(\lambda_k) - \delta_{k-1}(z)\right) \begin{bmatrix} 1 \\ -\delta_k(z) \end{bmatrix}.$$

Define

$$H(z) \equiv \prod_{k=1}^{n+1} \frac{z-\lambda_k}{1-\lambda_k z}, \quad \text{and} \quad G(z) \equiv \prod_{k=1}^{n+1} \left(\delta_{k-1}(\lambda_k) - \delta_{k-1}(z)\right).$$

Since $\mathbf{A}(z) = c_1 \mathbf{\Phi}(z)$, it follows that

$$G(z) = c_2 (1 - z\lambda_{n+1}) H(z) = c_2 H(z), \tag{A.5}$$

for some constant c_2, and that

$$\mathbf{\Phi}(z) \begin{bmatrix} 1 \\ -\delta_0(z) \end{bmatrix} = G(z) \begin{bmatrix} 1 \\ -\delta_{n+1}(z) \end{bmatrix}. \tag{A.6}$$

For a function $f(z)$, define the tilde operator as $\widetilde{f}(z) = f(1/z)$. Then,

$$\widetilde{\mathbf{A}}_k(z)' \mathbf{A}_k(z) = \left(\delta_{k-1}(\lambda_k)^2 + 1\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and, therefore,

$$\widetilde{\mathbf{\Phi}}(z)' \mathbf{\Phi}(z) = c_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{where} \quad c_3 = \prod_{k=1}^{n+1} \left(\delta_{k-1}(\lambda_k)^2 + 1\right).$$

Next, apply the tilde transformation to equation (A.6) to obtain

$$\widetilde{\mathbf{\Phi}}(z) \begin{bmatrix} 1 \\ -\widetilde{\delta}_0(z) \end{bmatrix} = \widetilde{G}(z) \begin{bmatrix} 1 \\ -\widetilde{\delta}_{n+1}(z) \end{bmatrix}.$$

Transposing and multiplying from the right with $\mathbf{\Phi}(z)$ yields

$$c_3 \begin{bmatrix} 1 & -\widetilde{\delta}_0(z) \end{bmatrix} = \begin{bmatrix} 1 & -\widetilde{\delta}_0(z) \end{bmatrix} \widetilde{\mathbf{\Phi}}(z)' \mathbf{\Phi}(z) = \widetilde{G}(z) \begin{bmatrix} 1 & -\tilde{\delta}_{n+1}(z) \end{bmatrix} \mathbf{\Phi}(z).$$

Together with the equation (A.6) we obtain four linear equations for the four entries of $\mathbf{\Phi}(z)$,

$$\begin{aligned}
\Phi_{11}(z) - \Phi_{12}(z)\delta_0(z) &= G(z) \\
\Phi_{21}(z) - \Phi_{22}(z)\delta_0(z) &= -\delta_{n+1}(z)G(z) \\
(\Phi_{11}(z) - \Phi_{21}(z)\widetilde{\delta}_{n+1}(z))\widetilde{G}(z) &= c_3 \\
(\Phi_{12}(z) - \Phi_{22}(z)\widetilde{\delta}_{n+1}(z))\widetilde{G}(z) &= -c_3\widetilde{\delta}_0(z).
\end{aligned}$$

The rank of the system is 3. Use the first, second and fourth equations to express $\Phi_{11}(z)$, $\Phi_{12}(z)$, $\Phi_{21}(z)$ in terms of $\Phi_{22}(z)$. The third equation does not allow to solve for $\Phi_{22}(z)$, rather it collapses to

$$\left(1 + \delta_{n+1}(z)\widetilde{\delta}_{n+1}(z)\right) G(z)\widetilde{G}(z) = \left(1 + \delta_0(z)\widetilde{\delta}_0(z)\right) c_3. \tag{A.7}$$

We can, now, determine $\Phi_{22}(z)$ from[1]

$$\Phi_{11}(z)\Phi_{22}(z) - \Phi_{12}(z)\Phi_{21}(z) = \det(\boldsymbol{\Phi}(z)) = c_3 H(z),$$

which implies

$$\Phi_{22}(z) = \frac{(G(z)\widetilde{\delta}_0(z)\delta_{n+1}(z) + \widetilde{G}(z)H(z))c_3}{(1+\delta_{n+1}(z)\widetilde{\delta}_{n+1}(z))G(z)\widetilde{G}(z)}.$$

Together with (A.7), this can be simplified to

$$\Phi_{22}(z) = \frac{G(z)\widetilde{\delta}_0(z)\delta_{n+1}(z) + \widetilde{G}(z)H(z)}{\delta_0(z)\widetilde{\delta}_0(z) + 1}.$$

Applying the tilde operation to equation (A.5) yields

$$\widetilde{G}(z) = c_2 \widetilde{H}(z).$$

Finally, it follows from the definition of $H(z)$ that $\widetilde{H}(z)H(z) = 1$, and therefore

$$\Phi_{22}(z) = \frac{c_2 H(z)\widetilde{\delta}_0(z)\delta_{n+1}(z) + c_2}{\delta_0(z)\widetilde{\delta}_0(z) + 1}. \qquad \square$$

A.4 Proof of Proposition 4.2

Suppose the equilibrium is invertible, then, the information structure is exogenous and equilibrium is unique. We can, therefore characterize it with a guess-and-verify approach. Guess

$$\pi_t = \alpha_0 \xi_t^s + \alpha_1 \xi_t^d + \alpha_2 \eta_t^s + \alpha_3 \eta_t^d, \qquad \text{and} \qquad c_t = \beta_0 \xi_t^s + \beta_1 \xi_t^d + \beta_2 \eta_t^s + \beta_3 \eta_t^d.$$

From the best-response functions, it follows that,

$$\pi_{it} = \mathbb{E}_{it}\left[\begin{array}{c} (\kappa(\beta_0+1) + (1-\theta)\alpha_0)\,\xi_t^s + (\kappa\beta_1 + (1-\theta)\alpha_1)\,\xi_t^d \\ + (\kappa\beta_2 + (1-\theta)\alpha_2)\,\eta_t^s + (\kappa\beta_3 + (1-\theta)\alpha_3)\,\eta_t^d \end{array}\right] + \delta\theta\mathbb{E}_{it}\left[\pi_{it+1}\right],$$

which implies

$$\begin{aligned} \pi_t = {} & \frac{\kappa(\beta_0+1) + (1-\theta)\alpha_0}{1-\theta\delta\rho}\mathbb{E}_t\left[\xi_t^s\right] + \frac{\kappa\beta_1 + (1-\theta)\alpha_1}{1-\theta\delta\rho}\mathbb{E}_t\left[\xi_t^d\right] \\ & + (\kappa\beta_2 + (1-\theta)\alpha_2)\,\mathbb{E}_t\left[\eta_t^s\right] + (\kappa\beta_3 + (1-\theta)\alpha_3)\,\mathbb{E}_t\left[\eta_t^d\right], \end{aligned}$$

and

$$c_{it} = \mathbb{E}_{it}\left[\begin{array}{c} \frac{\left(\mathtt{mpc}\,\beta_0 - \varsigma(1-\mathtt{mpc})\left(\phi_\pi-\rho\right)\alpha_0\right)}{1-(1-\mathtt{mpc})\rho}\xi_t^s + \frac{\left(\mathtt{mpc}\,\beta_1 - \varsigma(1-\mathtt{mpc})\left(1+\left(\phi_\pi-\rho\right)\alpha_1\right)\right)}{1-(1-\mathtt{mpc})\rho}\xi_t^d \\ + \left(\mathtt{mpc}\,\beta_2 - \varsigma\phi_\pi(1-\mathtt{mpc})\,\alpha_2\right)\eta_t^s + \left(\mathtt{mpc}\,\beta_3 - \varsigma\phi_\pi(1-\mathtt{mpc})\,\alpha_3\right)\eta_t^d \end{array}\right] + (1-\mathtt{mpc})\,\mathbb{E}_{it}\left[c_{it+1}\right],$$

[1]Notice that

$$\det(\mathbf{A}_k) = \left(1 + \delta_{k-1}(\lambda_k)^2\right)\frac{z-\lambda_k}{1-z\lambda_k}.$$

which implies

$$
\begin{aligned}
c_t = & \frac{\left(\mathtt{mpc}\,\beta_0 - \varsigma\left(1-\mathtt{mpc}\right)\left(\phi_\pi - \rho\right)\alpha_0\right)}{1-\left(1-\mathtt{mpc}\right)\rho}\mathbb{E}_t\left[\xi^s_t\right] + \frac{\left(\mathtt{mpc}\,\beta_1 - \varsigma\left(1-\mathtt{mpc}\right)\left(1+\left(\phi_\pi - \rho\right)\alpha_1\right)\right)}{1-\left(1-\mathtt{mpc}\right)\rho}\mathbb{E}_t\left[\xi^d_t\right] \\
& + \left(\mathtt{mpc}\,\beta_2 - \varsigma\phi_\pi\left(1-\mathtt{mpc}\right)\alpha_2\right)\mathbb{E}_t\left[\eta^s_t\right] + \left(\mathtt{mpc}\,\beta_3 - \varsigma\phi_\pi\left(1-\mathtt{mpc}\right)\alpha_3\right)\mathbb{E}_t\left[\eta^d_t\right].
\end{aligned}
$$

Let $\lambda = \frac{\tau}{1+\tau}$, then

$$
\mathbb{E}_t\left[\eta^s_t\right] = \lambda\eta^s_t, \text{ and } \mathbb{E}_t\left[\eta^d_t\right] = \lambda\eta^d_t.
$$

Matching coefficients, we obtain the following system of equations,

$$
\begin{aligned}
\alpha_0 &= \frac{\kappa\left(\beta_0+1\right)+\left(1-\theta\right)\alpha_0}{1-\theta\delta\rho}, \\
\alpha_1 &= \frac{\kappa\beta_1+\left(1-\theta\right)\alpha_1}{1-\theta\delta\rho}, \\
\left(\alpha_0+\alpha_2\right) &= \left(\left(\kappa\beta_2+\left(1-\theta\right)\alpha_2\right)+\frac{\kappa\left(\beta_0+1\right)+\left(1-\theta\right)\alpha_0}{1-\theta\delta\rho}\right)\lambda, \\
\left(\alpha_1+\alpha_3\right) &= \left(\left(\kappa\beta_3+\left(1-\theta\right)\alpha_3\right)+\frac{\kappa\beta_1+\left(1-\theta\right)\alpha_1}{1-\theta\delta\rho}\right)\lambda, \\
\beta_0 &= \frac{\left(\mathtt{mpc}\,\beta_0 - \varsigma\left(1-\mathtt{mpc}\right)\left(\phi_\pi-\rho\right)\alpha_0\right)}{1-\left(1-\mathtt{mpc}\right)\rho}, \\
\beta_1 &= \frac{\left(\mathtt{mpc}\,\beta_1 - \varsigma\left(1-\mathtt{mpc}\right)\left(1+\left(\phi_\pi-\rho\right)\alpha_1\right)\right)}{1-\left(1-\mathtt{mpc}\right)\rho}, \\
\left(\beta_0+\beta_2\right) &= \left(\left(\mathtt{mpc}\,\beta_2 - \varsigma\phi_\pi\left(1-\mathtt{mpc}\right)\alpha_2\right)+\frac{\left(\mathtt{mpc}\beta_0 - \varsigma\left(1-\mathtt{mpc}\right)\left(\phi_\pi-\rho\right)\alpha_0\right)}{1-\left(1-\mathtt{mpc}\right)\rho}\right)\lambda, \\
\left(\beta_1+\beta_3\right) &= \left(\left(\mathtt{mpc}\,\beta_3 - \varsigma\phi_\pi\left(1-\mathtt{mpc}\right)\alpha_3\right)+\frac{\left(\mathtt{mpc}\beta_1 - \varsigma\left(1-\mathtt{mpc}\right)\left(1+\left(\phi_\pi-\rho\right)\alpha_1\right)\right)}{1-\left(1-\mathtt{mpc}\right)\rho}\right)\lambda.
\end{aligned}
$$

Solving the system yields

$$
\begin{aligned}
\alpha_0 &= \frac{\kappa\left(1-\rho\right)}{\theta\left(1-\rho\right)\left(1-\delta\rho\right)+\kappa\varsigma\left(\phi_\pi-\rho\right)}, \\
\beta_0 &= \frac{\kappa\varsigma\left(\rho-\phi_\pi\right)}{\theta\left(1-\rho\right)\left(1-\delta\rho\right)+\kappa\varsigma\left(\phi_\pi-\rho\right)}, \\
\alpha_1 &= \frac{-\kappa\varsigma}{\theta\left(1-\rho\right)\left(1-\delta\rho\right)+\kappa\varsigma\left(\phi_\pi-\rho\right)}, \\
\beta_1 &= \frac{-\theta\varsigma\left(1-\delta\rho\right)}{\theta\left(1-\rho\right)\left(1-\delta\rho\right)+\kappa\varsigma\left(\phi_\pi-\rho\right)},
\end{aligned}
$$

and

$$\alpha_2 = \frac{\kappa(1-\lambda)\left(1-\rho-\mathtt{mpc}\lambda(1-\rho)-\kappa\varsigma\lambda\left(\phi_\pi-\rho\right)\right)}{\left(\theta(1-\rho)(1-\delta\rho)+\kappa\varsigma\left(\phi_\pi-\rho\right)\right)\left(\lambda(1-\theta)(1-\mathtt{mpc}\lambda)-\kappa\varsigma\phi_\pi\lambda^2(1-\mathtt{mpc})-(1-\mathtt{mpc}\lambda)\right)},$$

$$\beta_2 = \frac{\kappa\varsigma(1-\lambda)\left(\mathtt{mpc}\phi_\pi\lambda(1-\rho)-\left(\phi_\pi-\rho\right)(1+\theta\lambda)-\rho\lambda\left(1-\phi_\pi\right)\right)}{\left(\theta(1-\rho)(1-\delta\rho)+\kappa\varsigma\left(\phi_\pi-\rho\right)\right)\left(\lambda(1-\theta)(1-\mathtt{mpc}\lambda)-\kappa\varsigma\phi_\pi\lambda^2(1-\mathtt{mpc})-(1-\mathtt{mpc}\lambda)\right)},$$

$$\alpha_3 = \frac{-\kappa\varsigma(1-\lambda)((\theta-\mathtt{mpc})\lambda+1-\theta\delta\rho\lambda)}{\left(\theta(1-\rho)(1-\delta\rho)+\kappa\varsigma\left(\phi_\pi-\rho\right)\right)\left(\lambda(1-\theta)(1-\mathtt{mpc}\lambda)-\kappa\varsigma\phi_\pi\lambda^2(1-\mathtt{mpc})-(1-\mathtt{mpc}\lambda)\right)},$$

$$\beta_3 = \frac{\varsigma(\lambda-1)\left(\theta(1-\delta\rho)-\theta\lambda(1-\theta)(1-\delta\rho)-\kappa\varsigma\phi_\pi\lambda(1-\mathtt{mpc})\right)}{\left(\theta(1-\rho)(1-\delta\rho)+\kappa\varsigma\left(\phi_\pi-\rho\right)\right)\left(\left(\lambda(1-\theta)(1-\mathtt{mpc}\lambda)-\kappa\varsigma\phi_\pi\lambda^2(1-\mathtt{mpc})-(1-\mathtt{mpc}\lambda)\right)\right)}.$$

This solution validates the guess. Next, to guarantee that the equilibrium is indeed invertible, we need the following system to be invertible,

$$\begin{bmatrix} \pi_t \\ c_t \end{bmatrix} = \begin{bmatrix} \frac{\alpha_0}{1-\rho L}+\alpha_2 & \frac{\alpha_1}{1-\rho L}+\alpha_3 \\ \frac{\beta_0}{1-\rho L}+\beta_2 & \frac{\beta_1}{1-\rho L}+\beta_3 \end{bmatrix} \begin{bmatrix} \eta_t^s \\ \eta_t^d \end{bmatrix}.$$

For that, we need both roots of the determinant,

$$\Delta(L) = \left(\frac{\alpha_0}{1-\rho L}+\alpha_2\right)\left(\frac{\beta_1}{1-\rho L}+\beta_3\right)-\left(\frac{\alpha_1}{1-\rho L}+\alpha_3\right)\left(\frac{\beta_0}{1-\rho L}+\beta_2\right),$$

to be outside the unit circle. The roots are given by

$$r_1 \equiv -\frac{\tau(1+\theta-\mathtt{mpc})+\tau\sqrt{(1-\theta-\mathtt{mpc})^2-4(1-\mathtt{mpc})(\theta+\kappa\varsigma\phi_\pi)}}{2\rho},$$

$$r_2 \equiv -\frac{\tau(1+\theta-\mathtt{mpc})-\tau\sqrt{(1-\theta-\mathtt{mpc})^2-4(1-\mathtt{mpc})(\theta+\kappa\varsigma\phi_\pi)}}{2\rho}.$$

If $(1-\theta-\mathtt{mpc})^2-4(1-\mathtt{mpc})(\theta+\kappa\varsigma\phi_\pi)<0$, the roots are complex and their magnitude is above 1 if

$$\phi_\pi > \frac{\rho^2-\theta(1-\mathtt{mpc})\tau^2}{\kappa\varsigma(1-\mathtt{mpc})\tau^2}.$$

Otherwise, the roots are real and $|r_1|>|r_2|$. So we need to show that $|r_2|>1$, and for that we need to consider two cases: If $\tau(1+\theta-\mathtt{mpc})<2\rho$, then $|r_2|$ is always less than 1, otherwise, it is a necessary and sufficient condition that

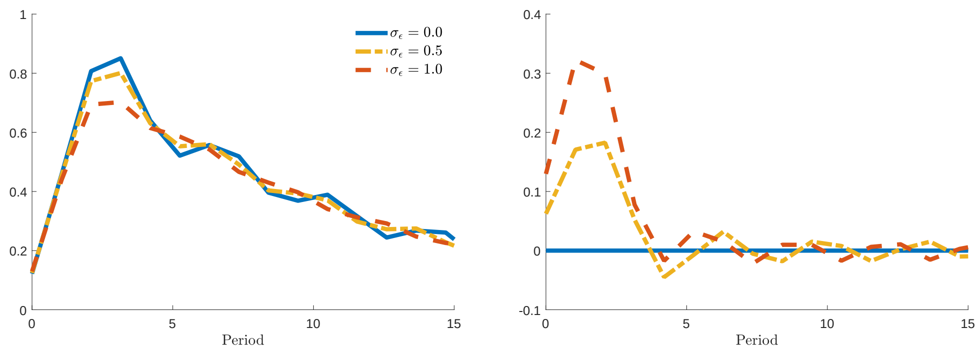
$$\phi_\pi > \frac{\tau\rho(1+\theta-\mathtt{mpc})-\rho^2-\theta(1-\mathtt{mpc})\tau^2}{\kappa\varsigma(1-\mathtt{mpc})\tau^2}.$$

B. Additional Noise in Exogenous Signals

We illustrate the point that our baseline result serves as a useful benchmark for other environments with non-square information. Consider the following modification of the exogenous signal while maintaining the assumption about perfect price observation,

$$x_{it} = \xi_t + u_{it} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2).$$

Here, the ϵ_t is a common noise that affects the signals of all agents. As a result, the price process is a function of both the monetary shock and the common noise shock. As σ_ϵ approaches zero, this economy returns to our baseline economy with a single aggregate shock. Figure 1 below displays the responses of prices to the fundamental shock η_t and to the common noise ϵ_t. When σ_ϵ is positive but relatively small, the price dynamics still exhibits an oscillatory pattern. When σ_ϵ is relatively large, the behavior of the model resembles the one with only exogenous signals, as in Angeletos and La'O (2010).



(a) shock to η_t (b) shock to ϵ_t

Figure 1: Responses of Prices to Fundamental and Noise Shocks

Parameters: $\rho = 0.9$, $\alpha = 0.9$, and $\tau = 1$.

C. Extensions

In this appendix we show that the main insights developed earlier extend to significantly more involved information structures, allowing for fundamentals that can follow any stochastic process, and multiple public and private signals with noise that also follows arbitrary processes. We also extend the results to environments with more sophisticated linkages among agents featuring forward and backward complementarities, and multivariate systems that can be viewed as a network game with incomplete information.

C.1 General Information Structure

To facilitate the analysis, we switch to a more general notation, since the applications of the following results encompass a wide range of settings that may differ from the one considered in the monetary model analyzed above.

Best Response. Denote agent i's action in period t by a_{it}. Their best response function is given by

$$a_{it} = \varphi \mathbb{E}_{it} [\xi_{it}] + \alpha \mathbb{E}_{it} [a_t], \tag{C.1}$$

where ξ_{it} denotes the individual fundamental, which can depend in a flexible way on the aggregate and idiosyncratic shocks, $\boldsymbol{\eta}_t$ and $\boldsymbol{u}_{it}$,

$$\xi_{it} = \boldsymbol{d}(L)\boldsymbol{\eta}_t + \boldsymbol{e}(L)\,\boldsymbol{u}_{it}, \quad \text{with} \quad \boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_\eta^2), \quad \text{and} \quad \boldsymbol{u}_{it} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_u^2).$$

The lag-operator polynomial vectors, $\boldsymbol{d}(L)$ and $\boldsymbol{e}(L)$, are assumed to have square-summable coefficients. We also assume that $\boldsymbol{d}(L)$ is not a constant vector to rule out fundamentals that are i.i.d. on common shocks; in which case the equilibrium is always invertible, and we rule out redundant shocks by assuming that $\boldsymbol{\Sigma}_\eta$ and $\boldsymbol{\Sigma}_u$ have full rank.

Information Structure. Agents have perfect recall and, in each period t, observe three sets of signals: (1) the previous period aggregate action, a_{t-1}, (2) a vector of private signals, $\boldsymbol{x}_{it}$, (3) and a vector of public signals, $\boldsymbol{z}_t$, where

$$\boldsymbol{x}_{it} = \mathbf{A}(L)\boldsymbol{\eta}_t + \mathbf{B}(L)\boldsymbol{u}_{it}, \quad \text{and} \quad \boldsymbol{z}_t = \mathbf{C}(L)\boldsymbol{\eta}_t.$$

We make two assumptions: first, that there are as many common shocks as there are public signals (including a_{t-1}), that is $\dim(\boldsymbol{\eta}_t) = \dim(\boldsymbol{z}_t) + 1$; second, that the matrix $\mathbf{B}(L)$ is invertible. To see why we impose the first assumption, note that, when $\dim(\boldsymbol{\eta}_t) < \dim(\boldsymbol{z}_t)+1$, there are more public signals than common shocks and past aggregate fundamentals are always perfectly revealed independently of the degree of strategic complementarity or of informational friction; when $\dim(\boldsymbol{\eta}_t) > \dim(\boldsymbol{z}_t) + 1$, there are more common shocks than public signals and the system is non-invertible by construction, as in Section 3.5. The interesting case for a discussion about invertibility is the one in which $\dim(\boldsymbol{\eta}_t) = \dim(\boldsymbol{z}_t) + 1$. The second assumption is more standard, it essentially excludes the cases in which non-invertibility is due to exogenously imposed shock processes. If $\mathbf{B}(L) = \mathbf{B}_0$, which is usually the case in most information structures considered in the literature, then the invertibility of $\mathbf{B}(L)$ is only violated if $\mathbf{B}_0$ is singular or not square in which case one of the private signals is redundant. Moreover, the polynomial matrices, $\mathbf{A}(L)$, $\mathbf{B}(L)$ and $\mathbf{C}(L)$, must have square-summable coefficients.

Invertibility. In the economy studied in Section 3 there is only one aggregate shock, the monetary shock, and invertibility is obtained if and only if the equilibrium process for the price index does not contain an inside root. When there are multiple aggregate shocks, the equilibrium process can be expressed as

$$a_t = \boldsymbol{g}(L)\boldsymbol{\eta}_t.$$

Driven by more than one aggregate shock, the aggregate outcome by itself can no longer reveal all underlying states. The relevant question becomes whether the history of signals, taken altogether, contains sufficient information.

Formally, we define an equilibrium process to be invertible if the history of the public signals, $\{a^t, \boldsymbol{z}^t\}$, contains the same information as the common shocks. The following lemma provides the corresponding criterion for invertibility.

Lemma C.1. *If* $\det\left[\boldsymbol{g}(L) \;\; \mathbf{C}(L)\right]^\top$ *does not contain any inside root, the equilibrium is invertible. Then, the public signals and the aggregate outcomes perfectly aggregate information,*

$$\mathbb{E}\left[\boldsymbol{\eta}_t | a^t, \boldsymbol{z}^t\right] = \boldsymbol{\eta}_t, \quad \textit{and} \quad \mathbb{E}\left[\xi_{it} | a^t, \boldsymbol{z}^t, \boldsymbol{x}_i^t\right] = \xi_{it}. \tag{C.2}$$

Proof. By definition, an equilibrium process to be invertible if the history of the public signals, $\{a^t, \boldsymbol{z}^t\}$, contains the same information as the common shocks, $\boldsymbol{\eta}^t$. Next, notice that

$$a_t = \boldsymbol{g}(L)\boldsymbol{\eta}_t \quad \text{and} \quad \boldsymbol{z}_t = \mathbf{C}(L)\boldsymbol{\eta}_t$$

can be stacked into

$$[a_t \ \ z_t]^\top = [\boldsymbol{g}(L) \ \ \mathbf{C}(L)]^\top \boldsymbol{\eta}_t.$$

This is a multivariate ARMA process which is inverlible if $\det[\boldsymbol{g}(L) \ \ \mathbf{C}(L)]^\top$ does not contain any roots inside the unit circle (see Brockwell and Davis (2002), Section 7.4). □

This result generalizes Lemma 3.1 to a multivariate system. The exogenously specified signal structure and the endogenously determined equilibrium process jointly determine whether agents can perfectly infer past shocks in the economy. Note that, once aggregate shocks are known, idiosyncratic shocks are also known since $\mathbf{B}(L)$ has been assumed to be invertible.

With the general information structure, we can no longer provide a simple partition of the parameter space into invertible and non-invertible regions as in Proposition 3.1. However, the basic insight derived in Section 3 remains true.

Theorem C.1. *There exists $\overline{\alpha} \in (-1, 1)$ such that, if $\alpha > \overline{\alpha}$, the equilibrium is not invertible.*

Proof. See Appendix C.3. □

The exact threshold $\overline{\alpha}$, above which the equilibrium is not invertible, depends on the details of the information structure. Independent of these details, however, such a threshold always exists. As in Section 3, if the degree of strategic complementarity is high enough, the equilibrium is not invertible. Under this more general information structure, however, the aggregate action may contain information on the aggregate fundamental as well as the common noise. The degree of strategic complementarity affects the invertibility of the joint dynamics of aggregate actions and public signals.

There is a sense in which the information structure we set up in this section is too general. In the following example, we consider a simplified structure that encompasses many used in the literature. Directly, it allows for the introduction of public signals to the monetary model from Section 3. We also allow agents to have idiosyncratic fundamentals which they observe with arbitrary precision. Hence, it also encompasses the information structure in the business-cycle model from Angeletos and La'O (2010) (with the addition of the observation of past aggregate outcomes). With this simpler structure, we can characterize in more detail how public signals affect information aggregation with takeaways that are still broadly applicable.

Example: how public signals affect invertibility. Suppose the best response function is given by equation (C.1) with the individual fundamental, ξ_{it}, and the aggregate fundamental, ξ_t, satisfying

$$\xi_{it} = \xi_t + \omega_{it}, \quad \omega_{it} \sim \mathcal{N}(0, \tau_\omega^{-1}), \quad \text{and} \quad \xi_t = d(L)\eta_t, \quad \eta_t \sim \mathcal{N}(0, 1),$$

for some arbitrary $d(L)$ with the normalization $d_0 = 1$. For simplicity we set $\varphi = 1 - \alpha$. Every period, agent i observes last period's aggregate action, a_{t-1}, a private, and a public signal,

$$x_{it} = \xi_{it} + u_{it}, \quad u_{it} \sim \mathcal{N}(0, \tau_u^{-1}), \quad \text{and} \quad z_t = \xi_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \tau_\varepsilon^{-1}).$$

In this setup, we can establish the following proposition which provides an explicit condition for invertibility. The proof of it also serves as a sketch of the proof of Theorem C.1.

Proposition C.1. *The equilibrium is invertible if and only if every root of the function* $\Gamma(z)$

$$\Gamma(z) \equiv d(z) - \frac{\alpha\tau_u + \tau_\omega}{\tau_u + \tau_\omega + (1-\alpha)\,\tau_u\,(\tau_\varepsilon + \tau_\omega)} \tag{C.3}$$

lies outside the unit circle, which is not the case for α *high enough. In particular, if the fundamental follows an AR(1) process, with* $d(L) = 1/(1-\rho L)$*, it is necessary and sufficient for invertibility that*

$$\alpha < 1 - \frac{\rho(\tau_u + \tau_\omega)}{\tau_u(1 + \rho + \tau_\varepsilon + \tau_\omega)}. \tag{C.4}$$

Proof. Suppose the equilibrium is invertible. Then, agents can infer past aggregate shocks perfectly and the effect of common shocks on the aggregate outcome can only be transitory. It follows that, the impulse response of the aggregate outcome to any common shock only differs from that of the aggregate fundamental, ξ_t, on impact. Accordingly, the law of motion of the aggregate outcome can be expressed as

$$a_t = \underbrace{g_\eta \eta_t + g_\varepsilon \varepsilon_t}_{\text{impact effect}} + (d(L) - 1)\eta_t.$$

To verify if the equilibrium is indeed invertible, we need to check the condition in Lemma C.1, which, in this case, reduces to checking if any of the roots of

$$\Gamma(z) = d(z) - \frac{1 - g_\eta}{1 - g_\varepsilon}, \tag{C.5}$$

are inside the unit circle. The impact effects

$$g_\eta = \frac{(1-\alpha)\tau_u(1+\tau_\omega) + \tau_\varepsilon(\tau_\omega + \tau_u)}{(1-\alpha)\tau_u\tau_\omega + (1+\tau_\varepsilon)(\tau_u + \tau_\omega)}, \quad \text{and} \quad g_\varepsilon = \frac{\tau_\varepsilon(\tau_\omega + \alpha\tau_u)}{(1-\alpha)\tau_u\tau_\omega + (1+\tau_\varepsilon)(\tau_u + \tau_\omega)}, \tag{C.6}$$

can be obtained by solving a simple static forecasting problem. Then, equation (C.3) follows from substituting equation (C.6) into (C.5). Next notice that, as α increases towards 1, equation (C.3) converges to $d(z) - 1$ which has root zero, since $d_0 = 1$. Hence, since the roots of a polynomial are a continuous function of its coefficients, there exists some $\overline{\alpha} \in (-1, 1)$ such that, for all $\alpha > \overline{\alpha}$, this equation has an inside root and the equilibrium cannot be invertible. Moreover, condition (C.4) follows immediately from solving for the root of equation (C.3) and requiring it to be outside the unit circle. □

Two opposing forces. How does the precision of the public signal, τ_ϵ, affect invertibility? Note that in the proof of Proposition C.1, the root of evil that causes non-invertibility is that $\frac{1-g_\eta}{1-g_\epsilon} \to 0$ when the magnitude of α approaches to zero. In general, the equilibrium is more likely to be invertible, or $\Gamma(z)$ less likely to have an inside root when $\frac{1-g_\eta}{1-g_\epsilon}$ is relatively large. This observation suggests that intuitively, a higher τ_ϵ has the following two opposing effects:

1. The extra precision leads to a better estimate of the fundamental and a stronger response to the fundamental, a higher g_η. This tends to make the equilibrium invertible.
2. The response to the common noise, g_ε, also increase since agents rely relatively more on the public signal z_t. This makes the information content of the aggregate action a_{t-1} closer to that of the public signal z_t. Put differently, there is less differential information contained in the aggregate outcome in comparison with the public signal, and this tends to make the equilibrium non-invertible.

To further appreciate this last point, consider the special case in which $d(L) = 1/(1-\rho L)$ and $\tau_\omega \to \infty$. The information structure, then, reduces to the one from Section 3 with the addition of a public signal. In this case, the invertibility condition (C.4) becomes

$$\alpha < 1 - \frac{\rho}{\tau_u},$$

which is identical to condition (7). This may seem puzzling at first, as the precision of the public signal plays no role in determining invertibility. However, this is because the two forces discussed above exactly cancel each other. In the extreme, when τ_ϵ goes to infinity, agents can infer the fundamental almost perfectly using the public signal. On the other hand, they correspondingly discard their private signals, and the aggregate outcome contains no more information than the one already obtained with the public signal. These effects cancel, leaving open the possibility that the equilibrium is not invertible.

In contrast, when τ_ω is finite, agents always use their private signals to learn about their idiosyncratic fundamental. It follows that the aggregate outcome necessarily aggregates the information contained in private signals, which differentiates itself from the public signal z_t. The two forces do not cancel each other, and the precision of the public signal, τ_ε, does matter for the determination of invertibility.

C.2 Forward Complementarities

The best-response function in equation (C.1) only allows for static strategic complementarities, that is, agent i's action depends on the current aggregate action. Here, we extend the analysis to allow for arbitrary forward-looking complementarities, that is, agent i's action can depend on future aggregate actions or on their own future actions in a flexible way. We consider the following best-response function,

$$a_{it} = \varphi \mathbb{E}_{it}[\xi_{it}] + \alpha \mathbb{E}_{it}[a_t] + \mathbb{E}_{it}[\gamma(L) a_t] + \mathbb{E}_{it}[\beta(L) a_{it}], \tag{C.7}$$

where

$$\gamma(L) \equiv \sum_{k=1}^{\infty} \gamma_k L^{-k}, \quad \beta(L) \equiv \sum_{k=1}^{\infty} \beta_k L^{-k}, \quad \text{and} \quad |\alpha| + \|\gamma(L)\| + \|\beta(L)\| < 1.$$

and we impose a relatively weak condition on the parameters that guarantees existence of the equilibrium.[2]

Even though the model structure is more sophisticated, it turns out that when the equilibrium is invertible, the general best-response function in condition (C.7) collapses to the static best response in condition (C.1) with

[2]In these conditions, $\|\cdot\|$ denotes the operator norm.

a modified fundamental, as forward-looking higher-order expectations collapse to first-order expectations in this scenario. The following proposition formalizes the required transformation.

Proposition C.2. *If the equilibrium is invertible, then the actions under best response (C.7) are observationally equivalent to those under the following transformed best response*

$$a_{it} = \varphi \mathbb{E}_{it}[\widetilde{\xi}_{it}] + \alpha \mathbb{E}_{it}[a_t], \quad \textit{with} \quad \widetilde{\xi}_{it} \equiv \frac{1-\alpha}{1-\alpha-\gamma(L)-\beta(L)} \xi_{it}.$$

Proof. See Appendix C.4. □

With this transformation, Theorem C.1 can be applied to the general class of best-response functions described by condition (C.7). The fact that there are forward complementaries does not change the fact that there always exists a threshold level for the degree of *static* strategic complementarity, $\overline{\alpha}$, such that, if $\alpha \geq \overline{\alpha}$, the equilibrium is not invertible.

The effects of forward complementarities on the invertibility of the equilibrium, however, are not as simple as in the static case. In Appendix C.5 we provide an analysis of the invertibility condition with an ARMA (1,1) fundamental process, which is fairly complicated as it hinges on the interaction between the fundamental and the coordination structure. In Appendix C.6, we show how Theorem C.1 can be further extended to economies with both forward and backward complementarities which encompasses many environments considered in the DSGE literature.

C.3 Proof of Theorem C.1

Suppose that when $\alpha = 0$ the equilibrium is invertible, otherwise the result is trivial. Section C.3.1 characterizes the equilibrium assuming invertibility. Using this characterization, Section C.3.2 takes the limit as α increases to 1 and shows that in it the equilibrium cannot be invertible.

C.3.1 Solution Assuming Invertibility

Suppose that the equilibrium is invertible, then the information set of agent i in period t is given by $\mathcal{I}_{it} \equiv \{\mathbf{x}_{i\tau}, \mathbf{z}_\tau, \boldsymbol{\eta}_{\tau-1}, \boldsymbol{u}_{i\tau-1}\}_{\tau=-\infty}^{t}$. Moreover,

$$\mathbb{E}_{it}[\xi_{it}] = \mathbb{E}[\boldsymbol{d}(L)\boldsymbol{\eta}_t + \boldsymbol{e}(L)\boldsymbol{u}_{it}|\mathcal{I}_{it}] = (\boldsymbol{d}(L) - \boldsymbol{d}_0)\boldsymbol{\eta}_t + \boldsymbol{d}_0\mathbb{E}[\boldsymbol{\eta}_t|\mathcal{I}_{it}] + (\boldsymbol{e}(L) - \boldsymbol{e}_0)\boldsymbol{u}_{it} + \boldsymbol{e}_0\mathbb{E}[\boldsymbol{u}_{it}|\mathcal{I}_{it}],$$

$$\mathbb{E}_{it}[a_t] = \mathbb{E}[\boldsymbol{g}(L)\boldsymbol{\eta}_t|\mathcal{I}_{it}] = (\boldsymbol{g}(L) - \boldsymbol{g}_0)\boldsymbol{\eta}_t + \boldsymbol{g}_0\mathbb{E}[\boldsymbol{\eta}_t|\mathcal{I}_{it}].$$

Moreover, since

$$\mathbf{x}_{it} - (\mathbf{A}(L) - \mathbf{A}_0)\boldsymbol{\eta}_t - (\mathbf{B}(L) - \mathbf{B}_0)\boldsymbol{u}_{it} = \mathbf{A}_0\boldsymbol{\eta}_t + \mathbf{B}_0\boldsymbol{u}_{it}, \quad \text{and} \quad \mathbf{z}_t - (\mathbf{C}(L) - \mathbf{C}_0)\boldsymbol{\eta}_t = \mathbf{C}_0\boldsymbol{\eta}_t,$$

it follows that $\mathbf{A}_0\boldsymbol{\eta}_t + \mathbf{B}_0\boldsymbol{u}_{it}$ is a noisy signal about $\boldsymbol{\eta}_t$ and $\boldsymbol{u}_{it}$, and $\mathbf{C}_0\boldsymbol{\eta}_t$ is a noisy signal about $\boldsymbol{\eta}_t$, which allows us to calculate

$$\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right] = \begin{bmatrix} \boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top & \boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top \end{bmatrix} \begin{bmatrix} \mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top + \mathbf{B}_0\boldsymbol{\Sigma}_u^2\mathbf{B}_0^\top & \mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top \\ \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top & \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_0\boldsymbol{\eta}_t + \mathbf{B}_0\boldsymbol{u}_{it} \\ \mathbf{C}_0\boldsymbol{\eta}_t \end{bmatrix},$$

$$\mathbb{E}\left[\boldsymbol{u}_{it}|\mathcal{I}_{it}\right] = \begin{bmatrix} \boldsymbol{\Sigma}_u^2\mathbf{B}_0^\top & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top + \mathbf{B}_0\boldsymbol{\Sigma}_u^2\mathbf{B}_0^\top & \mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top \\ \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top & \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_0\boldsymbol{\eta}_t + \mathbf{B}_0\boldsymbol{u}_{it} \\ \mathbf{C}_0\boldsymbol{\eta}_t \end{bmatrix}.$$

Therefore,

$$a_{it} = \varphi\left(\left(\boldsymbol{d}(L) - \boldsymbol{d}_0\right)\boldsymbol{\eta}_t + \boldsymbol{d}_0\,\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right] + \left(\boldsymbol{e}(L) - \boldsymbol{e}_0\right)\boldsymbol{u}_{it} + \boldsymbol{e}_0\,\mathbb{E}\left[\boldsymbol{u}_{it}|\mathcal{I}_{it}\right]\right) + \alpha\left(\left(\boldsymbol{g}(L) - \boldsymbol{g}_0\right)\boldsymbol{\eta}_t + \boldsymbol{g}_0\,\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right]\right),$$

which can be reorganized as

$$a_{it} = \left[\varphi\left(\boldsymbol{d}(L) - \boldsymbol{d}_0\right) + \alpha\left(\boldsymbol{g}(L) - \boldsymbol{g}_0\right)\right]\boldsymbol{\eta}_t + \varphi\left(\boldsymbol{e}(L) - \boldsymbol{e}_0\right)\boldsymbol{u}_{it} + \left(\varphi\boldsymbol{d}_0 + \alpha\boldsymbol{g}_0\right)\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right] + \varphi\boldsymbol{e}_0\mathbb{E}\left[\boldsymbol{u}_{it}|\mathcal{I}_{it}\right].$$

Consistency requires, in particular, that

$$\boldsymbol{g}(L) = \varphi\left(\boldsymbol{d}(L) - \boldsymbol{d}_0\right) + \alpha\left(\boldsymbol{g}(L) - \boldsymbol{g}_0\right) + \left(\varphi\boldsymbol{d}_0 + \alpha\boldsymbol{g}_0\right)\boldsymbol{k}_1\boldsymbol{\Omega} + \varphi\boldsymbol{e}_0\boldsymbol{k}_2\boldsymbol{\Omega},$$

where

$$\boldsymbol{k}_1 \equiv \begin{bmatrix} \boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top & \boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top \end{bmatrix}, \quad \boldsymbol{k}_2 \equiv \begin{bmatrix} \boldsymbol{\Sigma}_u^2\mathbf{B}_0^\top & \mathbf{0} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\Omega} \equiv \begin{bmatrix} \mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top + \mathbf{B}_0\boldsymbol{\Sigma}_u^2\mathbf{B}_0^\top & \mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top \\ \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top & \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_0 \\ \mathbf{C}_0 \end{bmatrix}.$$

We can rewrite the equation above as

$$(1-\alpha)\,\boldsymbol{g}(L) = \varphi\,\boldsymbol{d}(L) + \left(\varphi\boldsymbol{d}_0 + \alpha\boldsymbol{g}_0\right)\left(\boldsymbol{k}_1\boldsymbol{\Omega} - \mathbf{I}\right) + \varphi\boldsymbol{e}_0\boldsymbol{k}_2\boldsymbol{\Omega}$$

It is useful to replace the lag operator in this equation with an arbitrary complex number z. Evaluating this equation at $z = 0$, for instance, implies the following equilibrium condition,

$$\boldsymbol{g}_0 = \varphi\left(\boldsymbol{d}_0\boldsymbol{k}_1 + \boldsymbol{e}_0\boldsymbol{k}_2\right)\boldsymbol{\Omega}\left(\mathbf{I} - \alpha\boldsymbol{k}_1\boldsymbol{\Omega}\right)^{-1}.$$

Notice that, using the block-matrix inversion formula, $\boldsymbol{g}_0$ can be rewritten as

$$\boldsymbol{g}_0 = \frac{\varphi}{1-\alpha}\boldsymbol{d}_0\mathbf{E} + \varphi\left(\boldsymbol{d}_0\left(\mathbf{I} - \mathbf{E}\right)\mathbf{D} + \boldsymbol{e}_0\,\mathbf{F}\right)\sum_{j=0}^{\infty}\alpha^j\left[\left(\mathbf{I} - \mathbf{E}\right)\mathbf{D}\left(\mathbf{I} - \mathbf{E}\right)\right]^j, \tag{C.8}$$

where

$$
\begin{aligned}
\mathbf{D} &\equiv \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top \left(\mathbf{A}_0 \left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top + \mathbf{B}_0 \boldsymbol{\Sigma}_u^2 \mathbf{B}_0^\top \right)^{-1} \mathbf{A}_0, \\
\mathbf{E} &\equiv \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \left(\mathbf{C}_0 \boldsymbol{\Sigma}_\eta^2 \mathbf{C}_0^\top \right)^{-1} \mathbf{C}_0, \\
\mathbf{F} &\equiv \boldsymbol{\Sigma}_u^2 \mathbf{B}_0^\top \left(\mathbf{A}_0 \left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top + \mathbf{B}_0 \boldsymbol{\Sigma}_u^2 \mathbf{B}_0^\top \right)^{-1} \mathbf{A}_0,
\end{aligned}
$$

and we have used the fact that $\mathbf{E}$ is idempotent. Finally, substituting the expression for $\boldsymbol{g}_0$ into the equation for $\boldsymbol{g}(z)$ we obtain

$$
\boldsymbol{g}(z) = \boldsymbol{g}_0 + \frac{\varphi}{1-\alpha} (\boldsymbol{d}(z) - \boldsymbol{d}_0).
$$

C.3.2 Taking the Limit as α Increases to 1

Let

$$
\boldsymbol{g}(z) = \frac{1-\alpha}{\varphi} \boldsymbol{g}_0 + \boldsymbol{d}(z) - \boldsymbol{d}_0,
$$

so that

$$
\boldsymbol{g}(z) = \frac{\varphi}{1-\alpha} \boldsymbol{g}(z),
$$

and notice that this is well defined for all $\alpha < 1$ and that, if $\lim_{\alpha \to 1^-} \det \begin{bmatrix} \mathbf{C}(z) & \boldsymbol{g}(z) \end{bmatrix}^\top$ has an inside root, then there exists $\alpha < 1$ high enough such that $\det \begin{bmatrix} \mathbf{C}(z) & \boldsymbol{g}(z) \end{bmatrix}^\top$ is well defined and has an inside root. It is, in fact, sufficient to show that

$$
\lim_{\alpha \to 1^-} \det \begin{bmatrix} \mathbf{C}_0 & \boldsymbol{g}_0 \end{bmatrix}^\top = 0.
$$

Accordingly, using equation (C.8) we have that

$$
\lim_{\alpha \to 1^-} \boldsymbol{g}_0 = \lim_{\alpha \to 1^-} \frac{1-\alpha}{\varphi} \boldsymbol{g}_0 = \boldsymbol{d}_0 \mathbf{E} + \lim_{\alpha \to 1^-} (1-\alpha) \left(\boldsymbol{d}_0 \left(\mathbf{I} - \mathbf{E} \right) \mathbf{D} + \boldsymbol{e}_0 \mathbf{F} \right) \sum_{j=0}^{\infty} \alpha^j \left[\left(\mathbf{I} - \mathbf{E} \right) \mathbf{D} \left(\mathbf{I} - \mathbf{E} \right) \right]^j .
$$

In order to proceed, it is useful to establish the following lemma.

Lemma C.2. *The matrix* $\left(\mathbf{I} - \mathbf{E} \right) \mathbf{D} \left(\mathbf{I} - \mathbf{E} \right)$ *has all eigenvalues in* $[0, 1)$.

Proof. Notice that $\left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta^2$ is symmetric, so that $\left(\left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta^2 \right)^\top = \left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta^2$. Let $\mathbf{M} = \mathbf{A}_0 \left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta$ and $\mathbf{N} = \mathbf{B}_0 \boldsymbol{\Sigma}_u$, then it follows that

$$
\begin{aligned}
\mathbf{M}\mathbf{M}^\top &= \mathbf{A}_0 \left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta \left(\left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta \right)^\top \mathbf{A}_0^\top \\
&= \mathbf{A}_0 \left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta \left(\left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta^2 \boldsymbol{\Sigma}_\eta^{-1} \right)^\top \mathbf{A}_0^\top \\
&= \mathbf{A}_0 \left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta \boldsymbol{\Sigma}_\eta^{-1} \left(\left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta^2 \right)^\top \mathbf{A}_0^\top \\
&= \mathbf{A}_0 \left(\mathbf{I} - \mathbf{E} \right) \boldsymbol{\Sigma}_\eta^2 \mathbf{A}_0^\top .
\end{aligned}
$$

Therefore

$$
\left(\mathbf{I} - \mathbf{E} \right) \mathbf{D} \left(\mathbf{I} - \mathbf{E} \right) = \boldsymbol{\Sigma}_\eta \mathbf{M}^\top \left(\mathbf{M}\mathbf{M}^\top + \mathbf{N}\mathbf{N}^\top \right)^{-1} \mathbf{M} \boldsymbol{\Sigma}_\eta^{-1},
$$

so that $(\mathbf{I}-\mathbf{E})\,\mathbf{D}\,(\mathbf{I}-\mathbf{E})$ is similar to $\mathbf{M}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1}\mathbf{M}$. Let $n\equiv\dim(\boldsymbol{u}_{it})$ and $m\equiv\dim(\boldsymbol{\eta}_t)$, then, it follow that $\mathbf{N}$ is $n\times n$ and $\mathbf{M}$ is $m-1\times m$. If $n\geq m$,

$$\text{spectrum}\left(\mathbf{M}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1}\mathbf{M}\right)\cup\{0_1,\ldots,0_{n-m}\}=\text{spectrum}\left(\mathbf{M}\mathbf{M}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1}\right)$$

while if $n\leq m$,

$$\text{spectrum}\left(\mathbf{M}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1}\mathbf{M}\right)=\text{spectrum}\left(\mathbf{M}\mathbf{M}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1}\right)\cup\{0_1,\ldots,0_{m-n}\}.$$

The matrix $\mathbf{M}\mathbf{M}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1}$ is the product of a positive semi-definite with a positive definite matrix, so must have positive eigenvalues while $\mathbf{N}\mathbf{N}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1}$ is the product of two positive definite matrices, and, therefore, has strictly positive eigenvalues. Finally, since

$$\mathbf{M}\mathbf{M}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1}=\mathbf{I}-\mathbf{N}\mathbf{N}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1},$$

it follows that $\mathbf{M}\mathbf{M}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1}$ must have eigenvalues lower than 1. Hence,

$$\begin{aligned}\text{spectrum}\left((\mathbf{I}-\mathbf{E})\,\mathbf{D}\,(\mathbf{I}-\mathbf{E})\right)&=\text{spectrum}\left(\mathbf{M}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1}\mathbf{M}\right)\\&=\text{spectrum}\left(\mathbf{M}\mathbf{M}^\top\left(\mathbf{M}\mathbf{M}^\top+\mathbf{N}\mathbf{N}^\top\right)^{-1}\right)\subset[0,1).\end{aligned}$$

□

The lemma implies that

$$\sum_{j=0}^{\infty}\alpha^j\left[(\mathbf{I}-\mathbf{E})\,\mathbf{D}\,(\mathbf{I}-\mathbf{E})\right]^j$$

is well defined and finite, so that

$$\lim_{\alpha\to1^-}(1-\alpha)\left(\boldsymbol{d}_0\,(\mathbf{I}-\mathbf{E})\,\mathbf{D}+e_0\mathbf{F}\right)\sum_{j=0}^{\infty}\alpha^j\left[(\mathbf{I}-\mathbf{E})\,\mathbf{D}\,(\mathbf{I}-\mathbf{E})\right]^j=\mathbf{0}.$$

Therefore,

$$\lim_{\alpha\to1^-}\boldsymbol{g}_0=\boldsymbol{d}_0\mathbf{E},$$

and, using the definition of $\mathbf{E}$,

$$\boldsymbol{g}_0=\boldsymbol{d}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top\left(\mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top\right)^{-1}\mathbf{C}_0=\boldsymbol{a}\mathbf{C}_0,$$

for some vector $\boldsymbol{a}$. Finally, notice that

$$\det\begin{bmatrix}\mathbf{C}_0\\\boldsymbol{g}_0\end{bmatrix}=\det\begin{bmatrix}\mathbf{C}_0\\\boldsymbol{a}\mathbf{C}_0\end{bmatrix}=0,$$

which implies that $z=0$ is a root of $\det\begin{bmatrix}\mathbf{C}(z) & \boldsymbol{g}(z)\end{bmatrix}^\top$ and, therefore, $\begin{bmatrix}\mathbf{C}(L) & \boldsymbol{g}(L)\end{bmatrix}^\top$ is not invertible for α close enough to one (from below).

C.4 Proof of Proposition C.2

The following lemma establishes a type of law of iterated expectations.

Lemma C.3. *If $\mathcal{I}_{it} \supseteq \{\boldsymbol{\eta}_\tau, \boldsymbol{u}_{i\tau}\}_{\tau=-\infty}^{t-1}$, then, for any stochastic variable $y_{i,t+j} = \boldsymbol{f}(L)\boldsymbol{\eta}_{t+j} + \boldsymbol{g}(L)\boldsymbol{u}_{i,t+j} = \sum_{s=0}^{\infty} \boldsymbol{f}_s \boldsymbol{\eta}_{t+j-s} + \sum_{s=0}^{\infty} \boldsymbol{g}_s \boldsymbol{u}_{i,t+j-s}$,*

$$\mathbb{E}_{it}[\overline{\mathbb{E}}_{t+k}[y_{i,t+j}]] = \mathbb{E}_{it}[y_{i,t+j}], \qquad \textit{for all } k \geq 1.$$

Proof. Let $\boldsymbol{f}_s \equiv \mathbf{0}$, and $\boldsymbol{g}_s \equiv \mathbf{0}$, for all $s < 0$ and note that

$$\mathbb{E}_{i,t+k}[y_{i,t+j}] = \sum_{s=1}^{\infty}(\boldsymbol{f}_{j-k+s}\boldsymbol{\eta}_{t+k-s} + \boldsymbol{g}_{j-k+s}\boldsymbol{u}_{i,t+k-s}) + \boldsymbol{f}_{j-k}(\mathbf{H}_\eta \boldsymbol{\eta}_{t+k} + \mathbf{H}_u \boldsymbol{u}_{i,t+k}) + \boldsymbol{g}_{j-k}(\mathbf{P}_\eta \boldsymbol{\eta}_{t+k} + \mathbf{P}_u \boldsymbol{u}_{i,t+k}),$$

where we have used the fact that: (1) $\mathbb{E}_{i,t+k}[\boldsymbol{\eta}_{t+j}] = \mathbb{E}_{i,t+k}[\boldsymbol{u}_{i,t+j}] = \mathbf{0}$, for $j > k$; (2) $\mathbb{E}_{i,t+k}[\boldsymbol{\eta}_{t+j}] = \boldsymbol{\eta}_{t+j}$, and $\mathbb{E}_{i,t+k}[\boldsymbol{u}_{i,t+j}] = \boldsymbol{u}_{i,t+j}$, for $j < k$; and (3) $\mathbb{E}_{i,t+k}[\boldsymbol{\eta}_{t+k}] = \mathbf{H}_\eta \boldsymbol{\eta}_{t+k} + \mathbf{H}_u \boldsymbol{u}_{it+k}$, and $\mathbb{E}_{i,t+k}[\boldsymbol{u}_{i,t+k}] = \mathbf{P}_\eta \boldsymbol{\eta}_{t+k} + \mathbf{P}_u \boldsymbol{u}_{it+k}$, for some constant matrices $\mathbf{H}_\eta$, $\mathbf{H}_u$, $\mathbf{P}_\eta$, and $\mathbf{P}_u$.

In aggregate,

$$\overline{\mathbb{E}}_{t+k}[y_{i,t+j}] = \sum_{s=1}^{\infty} \boldsymbol{f}_{j-k+s}\boldsymbol{\eta}_{t+k-s} + \boldsymbol{f}_{j-k}\mathbf{H}_\eta \boldsymbol{\eta}_{t+k} + \boldsymbol{g}_{j-k}\mathbf{P}_\eta \boldsymbol{\eta}_{t+k}.$$

Consider agent i's the inference in period t,

$$\mathbb{E}_{it}\left[\overline{\mathbb{E}}_{t+k}[y_{i,t+j}]\right] = \sum_{s=1}^{\infty} \boldsymbol{f}_{j-s}\boldsymbol{\eta}_{t-s} + \boldsymbol{f}_j(\mathbf{H}_\eta \boldsymbol{\eta}_t + \mathbf{H}_u \boldsymbol{u}_{it}) = \mathbb{E}_{it}[y_{i,t+j}],$$

where the last equality follows from the same three facts listed above (with $k = 0$). □

Next, substituting the best-response (C.7) into itself, using the law of iterated expectations, we get

$$a_{it} = \mathbb{E}_{it}[\varphi\xi_{it} + (\alpha + \gamma(L))a_t + \beta(L)(\varphi\xi_{it} + (\alpha + \gamma(L))a_t + \beta(L)a_{it})],$$

and iterating on this procedure, using the fact that $\|\beta(L)\| < 1$ in the operator norm, leads to

$$a_{it} = \mathbb{E}_{it}[\varphi\widehat{\xi}_{it} + (\alpha + \kappa(L))a_t], \quad \text{where} \quad \widehat{\xi}_{it} \equiv \frac{1}{1-\beta(L)}\xi_{it} \quad \text{and} \quad \kappa(L) \equiv \frac{\gamma(L) + \alpha\beta(L)}{1-\beta(L)}. \tag{C.9}$$

Notice that $\kappa_0 = 0$. Aggregating implies

$$a_t = \overline{\mathbb{E}}_t[\varphi\widehat{\xi}_{it} + (\alpha + \kappa(L))a_t].$$

Multiplying both sides by $\kappa(L)$ and considering the inference of agent i in period t, we have that,

$$\mathbb{E}_{it}[\kappa(L)a_t] = \mathbb{E}_{it}\left[\kappa(L)\overline{\mathbb{E}}_t[\varphi\widehat{\xi}_{it} + (\alpha + \kappa(L))a_t]\right].$$

Since the equilibrium is invertible, $\mathcal{I}_{it} \supseteq \{\boldsymbol{\eta}_\tau, \boldsymbol{u}_{i\tau}\}_{\tau=-\infty}^{t-1}$, so that, using Lemma C.3, it follows that,

$$\mathbb{E}_{it}[\kappa(L)a_t] = \mathbb{E}_{it}[\varphi\kappa(L)\widehat{\xi}_{it} + \kappa(L)(\alpha + \kappa(L))a_t],$$

and, iterating on this procedure, using the fact that $||\alpha + \kappa(L)|| < 1$,[3] we obtain

$$\mathbb{E}_{it}[\kappa(L)a_t] = \mathbb{E}_{it}\left[\varphi\frac{\kappa(L)}{1-(\alpha+\kappa(L))}\widehat{\xi}_{it}\right].$$

The result follows from substituting this fact and the definitions of $\widehat{\xi}_{it}$ and $\kappa(L)$ into equation (C.9).

C.5 Forward Complementarity Example

Consider the following simple version of equation (C.7),

$$a_{it} = \varphi\mathbb{E}_{it}[\xi_{it}] + \alpha\mathbb{E}_{it}[a_t] + \gamma\mathbb{E}_{it}[a_{t+1}] + \beta\mathbb{E}_{it}[a_{it+1}],$$

and suppose that the equilibrium is invertible. Then, Proposition C.2 implies that this is equivalent to the static best response

$$a_{it} = \varphi\mathbb{E}_{it}[\widetilde{\xi}_{it}] + \alpha\mathbb{E}_{it}[a_t], \quad \text{with} \quad \widetilde{\xi}_{it} \equiv \frac{1-\alpha}{1-\alpha-(\gamma+\beta)L^{-1}}\,\xi_{it}.$$

As noted above, we can see that forward aggregate and individual complementarities, controlled by γ and β have interchangeable effects. This property allows us to simplify the analysis since we do not need to distinguish between the two types of forward complementarities focusing only on their sum.

Next, to be more concrete, suppose that ξ_{it} does not depend on idiosyncratic shocks (so we suppress the notation i) and follows an ARMA(1,1) process,

$$\xi_t = \rho\xi_{t-1} + \eta_t + \theta\eta_{t-1}, \quad \eta_t \sim \mathcal{N}(0,1),$$

and that, every period, agent i observes last period's aggregate action, a_{t-1}, and a private signal,

$$x_{it} = \xi_t + u_{it}, \quad u_{it} \sim \mathcal{N}(0,\tau_u^{-1}).$$

Then, we can establish the following corollary to Proposition C.2.

Corollary C.1. *The equilibrium is invertible if and only if*

$$\left|\frac{1-\alpha+\theta(\gamma+\beta)}{\rho(1-\alpha+\theta(\gamma+\beta))-(\rho+\theta)(1-\alpha H)}\right| > 1, \quad \textit{where} \quad H \equiv \frac{\tau_u}{1+\tau_u}. \tag{C.10}$$

Proof. It follows from Proposition C.2 that, for any $d(L)$ such that $\xi_t = d(L)\eta_t$, the law of motion for the aggregate action satisfies,

$$g(L) = \varphi\left[\frac{(1-\alpha)L}{(1-\alpha)L-(\gamma+\beta)}d(L)\right]_+ + \alpha(g(L)-g_0) + \alpha g_0 H.$$

where $[\cdot]_+$ denotes the annihilator operator, and $H \equiv \tau_u/(1+\tau_u)$. Let $\kappa \equiv (\gamma+\beta)/(1-\alpha)$, replace the lag operator

[3] To see this, notice that

$$||\alpha+\kappa(L)|| = \left\|\frac{\alpha+\gamma(L)}{1-\beta(L)}\right\| \leq ||\alpha+\gamma(L)||\left\|(1-\beta(L))^{-1}\right\| \leq |||\alpha|+\gamma(L)||||1-\beta(L)||^{-1} \leq \frac{\alpha+||\gamma(L)||}{1-||\beta(L)||} < 1,$$

where the first inequality follows from Cauchy-Schwarz, the second from the fact that, for any operator T, $||T||^{-1} \leq \left\|T^{-1}\right\|$, and the fourth from the assumptions that $||\beta(L)|| < 1$, and $|\alpha| + ||\gamma(L)|| + ||\beta(L)|| < 1$.

in this equation with an arbitrary complex number z, and rearrange to get

$$(1-\alpha)g(z) = \varphi \frac{zd(z) - \kappa d(\kappa)}{z-\kappa} + \alpha g_0(H-1).$$

We can obtain $g_0 = \varphi d(\kappa)(1-\alpha H)^{-1}$ by evaluating this equation at $z = 0$, and, then,

$$g(z) = g_0 + \frac{\varphi}{1-\alpha}\frac{d(z)-d(\kappa)}{z-\kappa}z.$$

Using the fact that $d(z) = (1+\theta z)/(1-\rho z)$, we obtain

$$g(z) = \frac{\varphi}{1-\rho\kappa}\left(\frac{1+\theta\kappa}{1-\alpha H} + \frac{1}{1-\alpha}\frac{(\rho+\theta)z}{1-\rho z}\right),$$

which has root

$$z^* = \frac{(1-\alpha)(1+\theta\kappa)}{\rho(1-\alpha)(1+\theta\kappa) - (\rho+\theta)(1-\alpha H)}.$$

The equilibrium is invertible if and only if this root to be outside the unit circle, that is $|z^*| > 1$. □

It is easy to see from equation (C.10) that the effect of forward complementarities, $\gamma+\beta$, on the invertibility of the equilibrium is ambiguous and depends both on the autoregressive and moving-average parameters, ρ and θ. To interpret this condition, it is useful to consider some particular cases. If, for instance, the fundamental follows an AR(1) process, with $\theta = 0$, the inequality simplifies to

$$\left|\frac{1-\alpha}{\alpha\rho(1-H)}\right| > 1,$$

and forward complementarities actually do not matter for invertibility.

To understand this, first let $[\cdot]_+$ denote the annihilator operator which sets negative powers of the lag operator to zero. Then, because the expected value of future shocks is always zero, we have that, for any stochastic variable $y_{i,t+j}$ and any j, $\mathbb{E}_{it}[y_{i,t+j}] = \mathbb{E}_{it}[[y_{i,t+j}]_+]$. When $\theta = 0$, we have that

$$[\widetilde{\xi}_t]_+ = \left[\frac{1-\alpha}{1-\alpha-(\gamma+\beta)L^{-1}}\frac{1}{1-\rho L}\eta_t\right]_+ = \frac{1-\alpha}{1-\alpha-\rho(\gamma+\beta)}\frac{1}{1-\rho L}\eta_t.$$

Thus, a change in $\gamma+\beta$ affects only the variance of the fundamental but not the autoregressive coefficient. Loosely speaking, when $\beta+\gamma$ increases, the agent puts relatively more weight on the next period fundamental. But since $\mathbb{E}_{it}[\xi_{t+1}] = \rho\mathbb{E}_{it}[\xi_t]$ this amounts to a proportional increase in the aggregate action in every period which does not affect invertibility.

Things are different if ξ_t follows an MA(1) process, that is, when $\rho = 0$ and $\theta \neq 0$. In that case, the inequality simplifies to

$$\left|\frac{1-\alpha+\theta(\gamma+\beta)}{\theta(1-\alpha H)}\right| > 1,$$

so that, if $\theta > 0$ (< 0) the equilibrium is non-invertible when $\gamma+\beta$ is low (high) enough.[4] In this case, we have

[4]More specifically, for $\theta > 0$, the equilibrium is non-invertible if $\gamma+\beta < (1-\alpha H) - (1-\alpha)/\theta$, and, for $\theta < 0$, if $\gamma+\beta > -(1-\alpha H) - (1-\alpha)/\theta$.

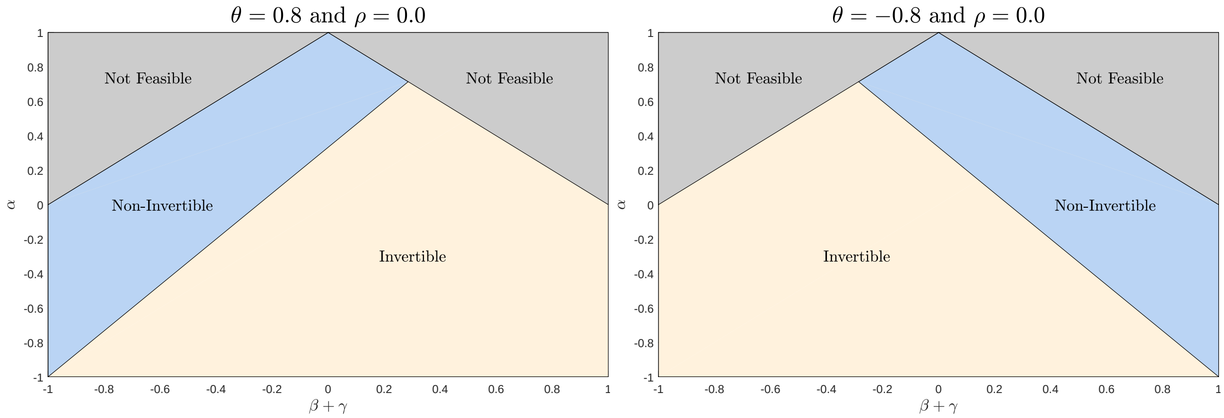


Figure 2: Regions of Invertibility with Forward-Looking Complementarities

The only free parameter, τ_u, is set to 1.

that

$$[\widetilde{\xi}_t]_+ = \left[\frac{1-\alpha}{1-\alpha-(\gamma+\beta)L^{-1}}(1+\theta L)\eta_t\right]_+ = \left(1 - \frac{\theta(\gamma+\beta)}{1-\alpha} + \theta L\right)\eta_t.$$

Here, it is useful to consider the response of the aggregate action, a_t, to a shock to the fundamental, η_t, in period $t = 0$, assuming the equilibrium is invertible. Also, for simplicity, suppose that $\theta > 0$. Then, an increase in $\gamma + \beta$ decreases the response of a_0 by an amount proportional to θ. On the other hand, it leaves a_1 unchanged since the impulse response of ξ_{t+k} for $k \geq 2$ is zero, so that forward-looking complementarities do not affect a_1 or the action in any further period. It follows that a_0/a_1 decreases, which reduces the signal-to-noise ratio and, therefore, the informativeness of the observation of a_0 to forecast a_1. This, in turn, makes it less likely that the equilibrium is indeed invertible.

Figure 2 shows how when the sign of the moving average parameter, θ, flips the effect of an increase in the degree of forward-looking complementarities, $\beta + \gamma$, on invertibility. It also illustrates, in accordance with Theorem C.1 and Proposition C.2, that it is always the case that for a high enough degree of *static* complementarity, α, the equilibrium is non-invertible.

C.6 Backward and Forward Complementarities

Section C.2 considers a best response function

$$a_{it} = \varphi \mathbb{E}_{it}[\xi_{it}] + \alpha \mathbb{E}_{it}[a_t] + \delta(L)\mathbb{E}_{it}[a_t] + \lambda(L)\mathbb{E}_{it}[a_{it}], \tag{C.11}$$

with forward looking complementarities, that is, assuming that $\delta(L)$ and $\lambda(L)$ are functions only of negative powers of the lag operator L. This section handles the cases in which backward complementarities as well. First, Section C.6.1 discusses the case with only static and backward complementarities. Then, Section C.6.2 deals with the case in which there are both backward and forward complementarities.

$$a_{it} = \varphi \mathbb{E}_{it}[\xi_{it}] + \alpha \mathbb{E}_{it}[a_t] + \delta(L)\mathbb{E}_{it}[a_t] + \lambda(L)\mathbb{E}_{it}[a_{it}],$$

C.6.1 Backward Complementarities

Consider the best response in equation C.11 with only backward complementarities, that is, such that $\delta(L)$ and $\lambda(L)$ only have positive powers of the lag polynomial. Since past aggregate actions have been assumed to be in agent's information sets, it immediately follows that

$$\mathbb{E}_{it}[a_{t-k}] = a_{t-k}, \qquad \text{for all } k \geq 1. \tag{C.12}$$

Assume that the perfect information equilibrium,

$$a_t = \frac{\varphi}{1-\alpha-\delta(L)-\lambda(L)}\,\xi_{it}$$

is well defined, that is, that $\|\alpha+\delta(L)+\lambda(L)\| < 1$ in the operator norm.

Proposition C.3. *The equilibrium is invertible with the best response*

$$a_{it} = \varphi\mathbb{E}_{it}[\xi_{it}] + \alpha\mathbb{E}_{it}[a_t],$$

if and only if it is invertible with the best response in equation C.11 with

$$\delta(L) = \sum_{k=1}^{\infty}\delta_k L^k, \quad \textit{and} \quad \lambda(L) = \sum_{k=1}^{\infty}\lambda_k L^k.$$

Proof. It follows from equation (C.12) that the best response with backward complementarities can be rewritten as

$$a_{it} = \varphi\mathbb{E}_{it}[\xi_{it}] + \alpha\mathbb{E}_{it}[a_t] + \delta(L)a_t + \lambda(L)a_{it}.$$

Analogously to the steps in the proof of Theorem C.1, we obtain the following consistency requirement for the law of motion of the aggregate action,

$$\boldsymbol{g}(L) = \varphi\left(\boldsymbol{d}(L)-\boldsymbol{d}_0\right) + \alpha\left(\boldsymbol{g}(L)-\boldsymbol{g}_0\right) + \left(\varphi\boldsymbol{d}_0+\alpha\boldsymbol{g}_0\right)\boldsymbol{k}_1\boldsymbol{\Omega} + \varphi\boldsymbol{e}_0\boldsymbol{k}_2\boldsymbol{\Omega} + (\delta(L)+\lambda(L))\boldsymbol{g}(L),$$

which can be rewritten as

$$(1-\alpha-\delta(L)-\lambda(L))\boldsymbol{g}(L) = \varphi\,\boldsymbol{d}(L) + \left(\varphi\boldsymbol{d}_0+\alpha\boldsymbol{g}_0\right)\left(\boldsymbol{k}_1\boldsymbol{\Omega}-\mathbf{I}\right) + \varphi\boldsymbol{e}_0\boldsymbol{k}_2\boldsymbol{\Omega}$$

Since $\delta_0 = \lambda_0 = 0$, we have that, just as in the proof of Theorem C.1,

$$\boldsymbol{g}_0 = \varphi\left(\boldsymbol{d}_0\boldsymbol{k}_1 + \boldsymbol{e}_0\boldsymbol{k}_2\right)\boldsymbol{\Omega}\left(\mathbf{I}-\alpha\boldsymbol{k}_1\boldsymbol{\Omega}\right)^{-1}.$$

and it follows that

$$\boldsymbol{g}(L) = \frac{(1-\alpha)\boldsymbol{g}_0 + \varphi(\boldsymbol{d}(L)-\boldsymbol{d}_0)}{1-\alpha-\delta(L)-\lambda(L)}.$$

Let

$$\boldsymbol{g}(z) = \frac{1-\alpha}{\varphi}\boldsymbol{g}_0 + \boldsymbol{d}(z) - \boldsymbol{d}_0,$$

so that

$$\boldsymbol{g}(z) = \frac{\varphi}{1-\alpha-\delta(z)-\lambda(z)}\boldsymbol{g}(z),$$

and notice that $\boldsymbol{g}(z)$ is the same as in the proof of Theorem C.1, so that, if the equilibrium is invertible in the static best response, it remains invertible with any feasible $\delta(L)$ and $\lambda(L)$ since, by assumption, $\|\alpha+\delta(L)+\lambda(L)\| < 1$. If it is non-invertible it remains non-invertible for the same reason. □

It follows that the result in Theorem C.1 immediately generalizes to settings with backward-looking complementarities. So that, regardless of these complementarities, if the static complementarity, α, is large enough the equilibrium is not invertible.

C.6.2 Interacting Backward and Forward Complementarities

Next, consider the following best-response function which encompasses most environments considered in the literature including, for instance, the Euler equation in a New-Keynesian model with capital,

$$a_{it} = \varphi\mathbb{E}_{it}[\xi_t] + \alpha\mathbb{E}_{it}[a_t] + \gamma\mathbb{E}_{it}[a_{t+1}] + \beta\mathbb{E}_{it}[a_{it+1}] + \delta\mathbb{E}_{it}[a_{t-1}] + \lambda\mathbb{E}_{it}[a_{it-1}].$$

Perfect Information Benchmark. It is easy to see that, if agents observe every shock up to the current period perfectly, the equilibrium must satisfy the following consistency requirement

$$\boldsymbol{g}(L) = \varphi\boldsymbol{d}(L) + \alpha\boldsymbol{g}(L) + (\gamma+\beta)\left(\frac{\boldsymbol{g}(L)-\boldsymbol{g}_0}{L}\right) + (\delta+\lambda)\boldsymbol{g}(L)L,$$

which, replacing the lag operator with an arbitrary complex number z, can be rewritten as

$$\left[-(\delta+\lambda)z^2 + (1-\alpha)z - (\gamma+\beta)\right]\boldsymbol{g}(z) = \varphi\boldsymbol{d}(z)z - (\gamma+\beta)\boldsymbol{g}_0.$$

In order for this equilibrium to exist and be unique, the polynomial on the left-hand side of this equation must have exactly one inside root, an assumption that we maintain throughout. By inside root we mean that the root is inside the unit circle in the complex plane. The right-hand side of the equation must be zero at any inside root of the polynomial at the left-hand side to avoid poles inside the unit circle. This condition is used to determine $\boldsymbol{g}_0$. With two outside roots, $\boldsymbol{g}_0$ is indeterminate so that there are multiple equilibria, and with two inside roots, $\boldsymbol{g}_0$ is over-determined so that, in general, an equilibrium does not exist. Let κ_1 and κ_2 be the inside and outside roots respectively, then, we would have that, the unique perfect-information equilibrium satisfies

$$\boldsymbol{g}(L) = \frac{\varphi}{\delta+\lambda}\frac{\kappa_2^{-1}}{1-\kappa_2^{-1}L}\frac{\boldsymbol{d}(L)L-\boldsymbol{d}(\kappa_1)\kappa_1}{L-\kappa_1}.$$

In what follows we only consider the set of parameters in which this perfect-information equilibrium exists and is unique, that is, such that $|\kappa_1| < 1$ and $|\kappa_2| > 1$. We refer to parameters that do not satisfies these conditions as infeasible. In this setup, we can establish the following result.

Theorem C.2. *Suppose that $\boldsymbol{e}(L) = \boldsymbol{e}$ and $\boldsymbol{B}(L) = \boldsymbol{B}$. For any $\omega_1, \omega_2 \neq 0$ such that $|\omega_1| < 1$ and $|\omega_2| < 1$, there exists $\epsilon > 0$ low enough such that if $\alpha = 1-(1+\omega_1\omega_2)\epsilon$, $\beta+\gamma = \omega_1\epsilon$, and $\delta+\lambda = \omega_2\epsilon$, the equilibrium is not invertible.*

Proof. The proof is presented in the next Section C.6.3. □

This theorem extends the result in Theorem C.1 for the case in which there are both forward and backward complementarities under the restriction that $\boldsymbol{e}(L) = \boldsymbol{e}$ and $\boldsymbol{B}(L) = \boldsymbol{B}$. This restriction is not particularly relevant since most environments considered in the literature do satisfy it. The theorem implies that there always a region in the space of feasible complementarity-parameters $(\alpha, \gamma, \beta, \delta, \lambda)$ such that the equilibrium is non-invertible. The reason why it is not enough to take the limit as the static degree of complementarity, α, increases to 1, as in Theorem C.1, is because, depending on the starting point, that might lead into an infeasible set of parameters. Hence, the limit must be taken in a careful enough way to guarantee that the region of non-invertibility is reached without violating feasibility.

C.6.3 Proof of Theorem C.2

This proof follows very similar steps to the ones in the proof of Theorem C.1, for clarity we closely follow the argument of that proof. Suppose that when $\alpha = 0$ the equilibrium is invertible, otherwise the result is trivial. We first characterize the equilibrium assuming invertibility. Using this characterization, we then take the appropriate limit and shows that, in it, the equilibrium cannot be invertible.

Solution Assuming Invertibility. Suppose that the equilibrium is invertible, then the information set of agent i in period t is given by $\mathcal{I}_{it} \equiv \{\boldsymbol{\eta}_{\tau-1}, \boldsymbol{u}_{i\tau-1}, \boldsymbol{x}_{i\tau}, \boldsymbol{z}_\tau\}_{\tau=-\infty}^{t}$. We guess (and verify below) that the individual policy function takes the form $a_{it} = \boldsymbol{g}(L)\boldsymbol{\eta}_t + \boldsymbol{h}\boldsymbol{u}_{it}$. Therefore,

$$\mathbb{E}_{it}\left[\xi_{it}\right] = \mathbb{E}\left[\boldsymbol{d}(L)\boldsymbol{\eta}_t + \boldsymbol{e}\boldsymbol{u}_{it}|\mathcal{I}_{it}\right] = \left(\boldsymbol{d}(L) - \boldsymbol{d}_0\right)\boldsymbol{\eta}_t + \boldsymbol{d}_0\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right] + \boldsymbol{e}\,\mathbb{E}\left[\boldsymbol{u}_{it}|\mathcal{I}_{it}\right],$$

$$\mathbb{E}_{it}\left[a_t\right] = \mathbb{E}\left[\boldsymbol{g}(L)\boldsymbol{\eta}_t|\mathcal{I}_{it}\right] = \left(\boldsymbol{g}(L) - \boldsymbol{g}_0\right)\boldsymbol{\eta}_t + \boldsymbol{g}_0\,\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right],$$

$$\mathbb{E}_{it}\left[a_{it+1}\right] = \mathbb{E}\left[\boldsymbol{g}(L)\boldsymbol{\eta}_{t+1} + \boldsymbol{h}\boldsymbol{u}_{it+1}|\mathcal{I}_{it}\right] = \left(\frac{\boldsymbol{g}(L) - (\boldsymbol{g}_0 + \boldsymbol{g}_1 L)}{L}\right)\boldsymbol{\eta}_t + \boldsymbol{g}_1\,\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right],$$

$$\mathbb{E}_{it}\left[a_{t+1}\right] = \mathbb{E}\left[\boldsymbol{g}(L)\boldsymbol{\eta}_{t+1}|\mathcal{I}_{it}\right] = \left(\frac{\boldsymbol{g}(L) - (\boldsymbol{g}_0 + \boldsymbol{g}_1 L)}{L}\right)\boldsymbol{\eta}_t + \boldsymbol{g}_1\,\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right],$$

$$\mathbb{E}_{it}\left[a_{it-1}\right] = \mathbb{E}\left[\boldsymbol{g}(L)\boldsymbol{\eta}_{t-1} + \boldsymbol{h}\boldsymbol{u}_{it-1}|\mathcal{I}_{it}\right] = \boldsymbol{g}(L)\boldsymbol{\eta}_{t-1} + \boldsymbol{h}\boldsymbol{u}_{it-1},$$

$$\mathbb{E}_{it}\left[a_{t-1}\right] = \mathbb{E}\left[\boldsymbol{g}(L)\boldsymbol{\eta}_{t-1}|\mathcal{I}_{it}\right] = \boldsymbol{g}(L)\boldsymbol{\eta}_{t-1}.$$

Moreover, since

$$\boldsymbol{x}_{it} - \left(\mathbf{A}(L) - \mathbf{A}_0\right)\boldsymbol{\eta}_t = \mathbf{A}_0\boldsymbol{\eta}_t + \mathbf{B}\boldsymbol{u}_{it}, \quad \text{and} \quad \boldsymbol{z}_t - \left(\mathbf{C}(L) - \mathbf{C}_0\right)\boldsymbol{\eta}_t = \mathbf{C}_0\boldsymbol{\eta}_t,$$

it follows that $\boldsymbol{x}_{it} - (\mathbf{A}(L) - \mathbf{A}_0)\,\boldsymbol{\eta}_t$ is a noisy signal about $\boldsymbol{\eta}_t$ and $\boldsymbol{u}_{it}$, and $\boldsymbol{z}_t - (\mathbf{C}(L) - \mathbf{C}_0)\,\boldsymbol{\eta}_t$ is a noisy signal about $\boldsymbol{\eta}_t$, which allows us to calculate

$$\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right] = \begin{bmatrix}\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top & \boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top\end{bmatrix}\begin{bmatrix}\mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top + \mathbf{B}\boldsymbol{\Sigma}_u^2\mathbf{B}^\top & \mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top \\ \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top & \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{A}_0\boldsymbol{\eta}_t + \mathbf{B}\boldsymbol{u}_{it} \\ \mathbf{C}_0\boldsymbol{\eta}_t\end{bmatrix},$$

$$\mathbb{E}\left[\boldsymbol{u}_{it}|\mathcal{I}_{it}\right] = \begin{bmatrix}\boldsymbol{\Sigma}_u^2\mathbf{B}^\top & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top + \mathbf{B}\boldsymbol{\Sigma}_u^2\mathbf{B}^\top & \mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top \\ \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top & \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{A}_0\boldsymbol{\eta}_t + \mathbf{B}\boldsymbol{u}_{it} \\ \mathbf{C}_0\boldsymbol{\eta}_t\end{bmatrix}.$$

Therefore,

$$\begin{aligned}a_{it} &= \varphi\left((\boldsymbol{d}(L) - \boldsymbol{d}_0)\,\boldsymbol{\eta}_t + \boldsymbol{d}_0\,\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right] + \boldsymbol{e}\,\mathbb{E}\left[\boldsymbol{u}_{it}|\mathcal{I}_{it}\right]\right) + \alpha\left((\boldsymbol{g}(L) - \boldsymbol{g}_0)\,\boldsymbol{\eta}_t + \boldsymbol{g}_0\,\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right]\right) \\ &\quad + \gamma\left(\left(\frac{\boldsymbol{g}(L) - (\boldsymbol{g}_0 + \boldsymbol{g}_1 L)}{L}\right)\boldsymbol{\eta}_t + \boldsymbol{g}_1\,\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right]\right) + \beta\left(\left(\frac{\boldsymbol{g}(L) - (\boldsymbol{g}_0 + \boldsymbol{g}_1 L)}{L}\right)\boldsymbol{\eta}_t + \boldsymbol{g}_1\,\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right]\right) \\ &\quad + (\delta + \lambda)\boldsymbol{g}(L)L\boldsymbol{\eta}_t + \lambda\boldsymbol{h}L\boldsymbol{u}_{it},\end{aligned}$$

which can be reorganized as

$$\begin{aligned}a_{it} &= \left[\varphi\left(\boldsymbol{d}(L) - \boldsymbol{d}_0\right) + \alpha\left(\boldsymbol{g}(L) - \boldsymbol{g}_0\right) + (\gamma + \beta)\left(\frac{\boldsymbol{g}(L) - (\boldsymbol{g}_0 + \boldsymbol{g}_1 L)}{L}\right) + (\delta + \lambda)\boldsymbol{g}(L)L\right]\boldsymbol{\eta}_t \\ &\quad + \lambda\boldsymbol{h}L\boldsymbol{u}_{it} + \left(\varphi\boldsymbol{d}_0 + \alpha\boldsymbol{g}_0 + (\gamma + \beta)\,\boldsymbol{g}_1\right)\mathbb{E}\left[\boldsymbol{\eta}_t|\mathcal{I}_{it}\right] + \varphi\boldsymbol{e}\mathbb{E}\left[\boldsymbol{u}_{it}|\mathcal{I}_{it}\right].\end{aligned}$$

Consistency requires, in particular, that

$$\begin{aligned}\boldsymbol{g}(L) &= \varphi\left(\boldsymbol{d}(L) - \boldsymbol{d}_0\right) + \alpha\left(\boldsymbol{g}(L) - \boldsymbol{g}_0\right) + (\gamma + \beta)\left(\frac{\boldsymbol{g}\,(L) - (\boldsymbol{g}_0 + \boldsymbol{g}_1 L)}{L}\right) + (\delta + \lambda)\boldsymbol{g}(L)L \\ &\quad + \left(\varphi\boldsymbol{d}_0 + \alpha\boldsymbol{g}_0 + (\gamma + \beta)\,\boldsymbol{g}_1\right)\boldsymbol{k}_1\boldsymbol{\Omega} + \varphi\boldsymbol{e}\boldsymbol{k}_2\boldsymbol{\Omega},\end{aligned}$$

where

$$\boldsymbol{k}_1 \equiv \begin{bmatrix}\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top & \boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top\end{bmatrix}, \quad \boldsymbol{k}_2 \equiv \begin{bmatrix}\boldsymbol{\Sigma}_u^2\mathbf{B}_0^\top & \mathbf{0}\end{bmatrix}, \quad \text{and} \quad \boldsymbol{\Omega} \equiv \begin{bmatrix}\mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top + \mathbf{B}_0\boldsymbol{\Sigma}_u^2\mathbf{B}_0^\top & \mathbf{A}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top \\ \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top & \mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{A}_0 \\ \mathbf{C}_0\end{bmatrix}.$$

The guess for the policy function can be verified by collecting the terms associated with the idiosyncratic shocks and noticing that they are zero for any period other than the current one. Thus, we can rewrite the equation above as

$$\begin{aligned}(1 - \alpha)\left(L - \frac{\gamma + \beta}{1 - \alpha} - \frac{\delta + \lambda}{1 - \alpha}L^2\right)\boldsymbol{g}(L) &= \varphi\,\boldsymbol{d}(L)L - (\gamma + \beta)\,\boldsymbol{g}_0 \\ &\quad + \left(\varphi\boldsymbol{d}_0 + \alpha\boldsymbol{g}_0 + (\gamma + \beta)\,\boldsymbol{g}_1\right)(\boldsymbol{k}_1\boldsymbol{\Omega} - \mathbf{I})\,L + \varphi\boldsymbol{e}\boldsymbol{k}_2\boldsymbol{\Omega}L \qquad \text{(C.13)}\end{aligned}$$

It is useful to replace the lag operator in this equation with an arbitrary complex number z. Evaluating this equation at different values of z implies conditions that allow for the characterization of the equilibrium.

Solving for $\boldsymbol{g}(z)$. The right-hand side of equation (C.13) must be equal to $\mathbf{0}$ when evaluated at the inside root, κ_1, of the second-order polynomial on the left-hand side of the equation; we denote the outside root by

κ_2.[5] Moreover, the equation must be consistent with the values of $\boldsymbol{g}_0$ and $\boldsymbol{g}_1$. Consistency at $z=0$, i.e. $\boldsymbol{g}(0)=\boldsymbol{g}_0$ is automatic. Next, set $z=\kappa_1$ to get

$$(1-\alpha)\,\boldsymbol{g}_0 = \varphi\boldsymbol{d}(\kappa_1) + (\varphi\boldsymbol{d}_0 + \alpha\boldsymbol{g}_0 + (\gamma+\beta)\,\boldsymbol{g}_1)\,(\boldsymbol{k}_1\boldsymbol{\Omega} - \mathbf{I}) + \varphi\boldsymbol{e}\boldsymbol{k}_2\boldsymbol{\Omega}.$$

It follows that

$$\frac{\boldsymbol{g}(z)-\boldsymbol{g}_0}{z} = -\frac{\varphi(\boldsymbol{d}(z)-\boldsymbol{d}(\kappa_1))}{(\delta+\gamma)(z-\kappa_1)(z-\kappa_2)} - \frac{\boldsymbol{g}_0}{z-\kappa_2},$$

and since, by definition,

$$\boldsymbol{g}_1 = \left.\frac{\boldsymbol{g}(z)-\boldsymbol{g}_0}{z}\right|_{z=0},$$

we obtain

$$\boldsymbol{g}_1 = \frac{\boldsymbol{g}_0}{\kappa_2} + \frac{\varphi\,(\boldsymbol{d}(\kappa_1)-\boldsymbol{d}_0)}{\gamma+\beta},$$

Putting these results together we obtain that

$$(\kappa_2 - z)\boldsymbol{g}(z) = \kappa_2\boldsymbol{g}_0 + \frac{\varphi}{\gamma+\beta}\frac{\boldsymbol{d}(z)-\boldsymbol{d}(\kappa_1)}{z-\kappa_1}z,$$

where

$$\boldsymbol{g}_0 = \varphi\,(\boldsymbol{d}(\kappa_1)\boldsymbol{k}_1 + \boldsymbol{e}\boldsymbol{k}_2)\,\boldsymbol{\Omega}\,(\mathbf{I} - (\alpha+(\delta+\lambda)\kappa_1)\boldsymbol{k}_1\boldsymbol{\Omega})^{-1}.$$

Notice that, using the block-matrix inversion formula, $\boldsymbol{g}_0$ can be rewritten as

$$\begin{aligned}\boldsymbol{g}_0 = &\frac{\varphi}{1-(\alpha+(\delta+\lambda)\kappa_1)}\boldsymbol{d}(\kappa_1)\mathbf{E}\\ &+\varphi\,(\boldsymbol{d}(\kappa_1)\,(\mathbf{I}-\mathbf{E})\,\mathbf{D} + \boldsymbol{e}\,\mathbf{F})\sum_{j=0}^{\infty}(\alpha+(\delta+\lambda)\kappa_1)^j\,[(\mathbf{I}-\mathbf{E})\,\mathbf{D}\,(\mathbf{I}-\mathbf{E})]^j\,,\end{aligned}\tag{C.14}$$

where

$$\begin{aligned}\mathbf{D} &\equiv \boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top\left(\mathbf{A}_0\,(\mathbf{I}-\mathbf{E})\,\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top + \mathbf{B}_0\boldsymbol{\Sigma}_u^2\mathbf{B}_0^\top\right)^{-1}\mathbf{A}_0,\\ \mathbf{E} &\equiv \boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top\left(\mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top\right)^{-1}\mathbf{C}_0,\\ \mathbf{F} &\equiv \boldsymbol{\Sigma}_u^2\mathbf{B}^\top\left(\mathbf{A}_0\,(\mathbf{I}-\mathbf{E})\,\boldsymbol{\Sigma}_\eta^2\mathbf{A}_0^\top + \mathbf{B}_0\boldsymbol{\Sigma}_u^2\mathbf{B}_0^\top\right)^{-1}\mathbf{A}_0,\end{aligned}$$

and we have used the fact that $\mathbf{E}$ is idempotent.

Taking the Appropriate Limit. Suppose that $\alpha = 1-(1+\omega_1\omega_2)\epsilon$, $\gamma+\beta=\omega_1\epsilon$, and $\delta+\lambda=\omega_2\epsilon$. It follows that, for all $\epsilon>0$,

$$\kappa_1 = \omega_1, \quad \kappa_2 = \omega_2^{-1}, \quad \text{and} \quad \alpha+(\delta+\lambda)\kappa_1 = 1-\epsilon.$$

[5]Explicitly,

$$\kappa_1 = \frac{1-\alpha-\sqrt{(1-\alpha)^2-4(\gamma+\beta)(\delta+\lambda)}}{2(\delta+\lambda)}, \quad \text{and} \quad \kappa_2 = \frac{1-\alpha+\sqrt{(1-\alpha)^2-4(\gamma+\beta)(\delta+\lambda)}}{2(\delta+\lambda)}.$$

To establish the claim, we consider the limit of

$$\boldsymbol{g}(z) = \frac{\varphi}{(1-\omega_2 z)\epsilon}\left(\frac{\omega_2}{\omega_1}\frac{\boldsymbol{d}(z)-\boldsymbol{d}(\omega_1)}{z-\omega_1}z + \frac{\epsilon}{\varphi}\boldsymbol{g}_0\right),$$

as ϵ decreases towards 0. Let

$$\boldsymbol{g}(z) = \frac{\omega_2}{\omega_1}\frac{\boldsymbol{d}(z)-\boldsymbol{d}(\omega_1)}{z-\omega_1}z + \frac{\epsilon}{\varphi}\boldsymbol{g}_0,$$

so that

$$\boldsymbol{g}(z) = \frac{\varphi}{(1-\omega_2 z)\epsilon}\boldsymbol{g}(z),$$

and notice that this is well defined for all $\epsilon > 0$ and that, if $\lim_{\epsilon\to 0^+}\det\left(\begin{bmatrix}\mathbf{C}(z) & \boldsymbol{g}(z)\end{bmatrix}^\top\right)$ has an inside root, then there exists ϵ low enough such that $\det\left(\begin{bmatrix}\mathbf{C}(z) & \boldsymbol{g}(z)\end{bmatrix}^\top\right)$ is well defined and has an inside root. Recall that $|\omega_2| < 1$. It is, in fact, sufficient to show that

$$\lim_{\epsilon\to 0^+}\det\left(\begin{bmatrix}\mathbf{C}(0) & \boldsymbol{g}_0\end{bmatrix}^\top\right) = 0.$$

Accordingly, using equation (C.14) we have that

$$\lim_{\epsilon\to 0^+}\boldsymbol{g}_0 = \lim_{\epsilon\to 0^+}\frac{\epsilon}{\varphi}\boldsymbol{g}_0 = \boldsymbol{d}(\omega_1)\mathbf{E} + \lim_{\epsilon\to 0^+}\epsilon\left(\boldsymbol{d}(\omega_1)(\mathbf{I}-\mathbf{E})\mathbf{D} + \boldsymbol{e}\mathbf{F}\right)\sum_{j=0}^{\infty}(1-\epsilon)^j\left[(\mathbf{I}-\mathbf{E})\mathbf{D}(\mathbf{I}-\mathbf{E})\right]^j.$$

It follows from Lemma C.2 that

$$\sum_{j=0}^{\infty}(1-\epsilon)^j\left[(\mathbf{I}-\mathbf{E})\mathbf{D}(\mathbf{I}-\mathbf{E})\right]^j$$

is well defined and finite, so that

$$\lim_{\epsilon\to 0^+}\epsilon\left(\boldsymbol{d}(\omega_1)(\mathbf{I}-\mathbf{E})\mathbf{D} + \boldsymbol{e}\mathbf{F}\right)\sum_{j=0}^{\infty}(1-\epsilon)^j\left[(\mathbf{I}-\mathbf{E})\mathbf{D}(\mathbf{I}-\mathbf{E})\right]^j = \mathbf{0}.$$

Therefore,

$$\lim_{\epsilon\to 0^+}\boldsymbol{g}_0 = \boldsymbol{d}(\omega_1)\mathbf{E}$$

and, using the definition of $\mathbf{E}$,

$$\boldsymbol{g}_0 = \boldsymbol{d}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top\left(\mathbf{C}_0\boldsymbol{\Sigma}_\eta^2\mathbf{C}_0^\top\right)^{-1}\mathbf{C}_0 = \boldsymbol{a}\mathbf{C}_0$$

for some vector $\boldsymbol{a}$. Finally, notice that

$$\det\begin{bmatrix}\mathbf{C}_0\\ \boldsymbol{g}_0\end{bmatrix} = \det\begin{bmatrix}\mathbf{C}_0\\ \boldsymbol{a}\mathbf{C}_0\end{bmatrix} = 0,$$

which implies that $z = 0$ is a root of $\det\begin{bmatrix}\mathbf{C}(z) & \boldsymbol{g}(z)\end{bmatrix}^\top$ and, therefore, $\begin{bmatrix}\mathbf{C}(L) & \boldsymbol{g}(L)\end{bmatrix}^\top$ is not invertible for ϵ close enough to zero.

D. Invertibility in New Keynesian Model

Firms: The New Keynesian Phillips Curve. The optimal reset price solves the following problem:

$$P^*_{i,t} = \arg\max_{P_{i,t}} \sum_{k=0}^{\infty} (\delta\theta)^k \, \mathbb{E}_{i,t} \left[Q_{t|t+k} \left(P_{i,t} Y_{i,t+k|t} - P_{t+k} mc_{t+k} Y_{i,t+k|t} \right) \right]$$

subject to the demand equation, $Y_{i,t+k} = \left(\frac{P_{i,t}}{P_{t+k}}\right)^{-\epsilon} Y_{t+k}$, where $Q_{t|t+k}$ is the stochastic discount factor between t and $t+k$, Y_{t+k} and P_{t+k} are, respectively, aggregate income and the aggregate price level in period $t+k$, $P_{i,t}$ is the firm's price, as set in period t, $Y_{i,t+k|t}$ is the firm's quantity in period $t+k$, conditional on not having changed the price since t, and mc_{t+k} is the real marginal cost in period $t+k$. The firm's discount factor is δ, and θ is the Calvo parameter (probability of not resetting price).

Taking the first-order condition and log-linearizing around a steady state with no shocks and zero inflation, we get the following, familiar, characterization of the optimal reset price:

$$p^*_{i,t} = (1-\delta\theta) \sum_{k=0}^{\infty} (\delta\theta)^k \, \mathbb{E}_{i,t} \left[mc_{t+k} + p_{t+k} \right]. \tag{D.1}$$

Suppose that firms observe the aggregate prices up to period $t-1$, that is, they observe p^{t-1}, then we can restate condition (D.1) as

$$p^*_{i,t} - p_{t-1} = (1-\delta\theta) \sum_{k=0}^{\infty} (\delta\theta)^k \, \mathbb{E}_{i,t} \left[mc_{t+k} \right] + \sum_{k=0}^{\infty} (\delta\theta)^k \, \mathbb{E}_{i,t} \left[\pi_{t+k} \right]. \tag{D.2}$$

Since only a fraction $1-\theta$ of the firms adjust their prices each period, the price level in period t is given by $p_t = (1-\theta) \int p^*_{i,t} \, \mathrm{d}i + \theta p_{t-1}$, and inflation is given by

$$\pi_t \equiv p_t - p_{t-1} = (1-\theta) \int \left(p^*_{i,t} - p_{t-1} \right) \mathrm{d}i.$$

Define the firm specific inflation rate to be

$$\pi_{i,t} \equiv (1-\theta) \left(p^*_{i,t} - p_{t-1} \right).$$

Then, it follows from equation (D.2) that

$$\pi_{i,t} = (1-\theta) \, \mathbb{E}_{i,t} \left[(1-\delta\theta) \, mc_t + \pi_t \right] + \delta\theta \mathbb{E}_{i,t} \left[(1-\theta) \sum_{k=0}^{\infty} (\delta\theta)^k \, \mathbb{E}_{i,t+1} \left[(1-\delta\theta) \, mc_{t+1+k} + \pi_{t+1+k} \right] \right],$$

and we obtain the following beauty contest game, which includes equation (11),

$$\pi_{i,t} = (1-\theta)(1-\delta\theta) \, \mathbb{E}_{i,t} \left[mc_t \right] + (1-\theta) \, \mathbb{E}_{i,t} \left[\pi_t \right] + \delta\theta \mathbb{E}_{i,t} \left[\pi_{i,t+1} \right], \quad \text{with } \pi_t = \int \pi_{i,t} \, \mathrm{d}i.$$

Households: The Dynamic IS. The consumer's problem is

$$\max_{\{C_{i,t},B_{i,t+1}\}} \mathbb{E}_{i,t}\left[\sum_{k=0}^{\infty}\beta^k \frac{C_{i,t+k}^{1-\frac{1}{\varsigma}}}{1-\frac{1}{\varsigma}}\right]$$

subject to

$$C_{i,t} + B_{i,t+1} = R_{t-1}B_{i,t} + Y_t,$$

where R_t and W denote real interest rates and income. At any state, the life-time budget constraint can be written as

$$\sum_{k=0}^{\infty}\frac{C_{i,t+k}}{\prod_{j=1}^{k}R_{t+j-1}} = R_{t-1}B_{i,t} + \sum_{k=0}^{\infty}\frac{Y_t}{\prod_{j=1}^{k}R_{t+j-1}},$$

which can be log-linearized into

$$\sum_{k=0}^{\infty}\beta^k c_{i,t+k} = \sum_{k=0}^{\infty}\beta^k y_{t+k}.$$

Combining this with the log-linearized version of the households' Euler equation,

$$c_{i,t} = \mathbb{E}_{i,t}[c_{i,t+1}] - \varsigma\mathbb{E}_{i,t}[r_t],$$

and using the market clearing condition, $y_t = c_t$, we obtain

$$c_{i,t} = -\varsigma\beta\sum_{k=0}^{\infty}\beta^k\mathbb{E}_{i,t}[r_{t+k}] + (1-\beta)\sum_{k=0}^{\infty}\beta^k\mathbb{E}_{i,t}[c_{t+k}].$$

Finally, notice this is implied by the following beauty-contest game,

$$c_{i,t} = -\varsigma\beta\mathbb{E}_{i,t}[r_t] + (1-\beta)\mathbb{E}_{i,t}[c_t] + \beta\mathbb{E}_{i,t}[c_{i,t+1}], \quad \text{with } c_t = \int c_{i,t}\,\mathrm{d}i.$$

Letting $\mathtt{mpc} = 1-\beta$ yields equation (12).

D.1 Invertibility: Proof of Proposition 4.1

Suppose that ξ_t follows an AR(1) process

$$\xi_t = \rho\xi_{t-1} + \eta_t + \theta\eta_{t-1}, \quad \text{with } \eta_t \sim N(0,1),$$

and that, besides past prices, firms only observe a private signal about it with precision τ_u,

$$x_{i,t} = \xi_t + u_{i,t}, \quad \text{with } u_{i,t} \sim N\left(0, \tau_u^{-1}\right).$$

Then, it follows from Corollary C.1 that for a best response

$$a_{i,t} = \varphi\mathbb{E}_{i,t}[\xi_t] + \alpha\mathbb{E}_{i,t}[a_t] + \gamma\mathbb{E}_{i,t}[a_{t+1}] + \beta\mathbb{E}_{i,t}[a_{i,t+1}]$$

the equilibrium is invertible if and only if

$$C(\alpha) \equiv \left|\frac{1-\alpha}{\rho\alpha(1-H)}\right| > 1, \quad \text{where} \quad H \equiv \frac{\tau_u}{1+\tau_u}.$$

For positive α and ρ, the relevant case here, it is easy to see that $C(\alpha)$ is a decreasing function. Hence, higher degrees of static strategic complementarity make it less likely that the equilibrium is invertible. Since this controlled by $1-\theta$ in the NKPC and $\texttt{mpc} \equiv 1-\beta$ in the Dynamic IS, invertibility is less likely for lower θ and higher $\texttt{mpc}$.

E. Additional Material for Section 4.3

E.1 Recovering Model IRFs via VAR

In Section 4.3.2, we remark on the comparability between the IRFs obtained by Angeletos, Collard, and Dellas (2020) from the data using a VAR that assumes invertibility and the IRFs from the model in which output is non invertible. To address this, we make two points here: (1) Even if a multivariate VAR system is invertible, the conditional response of a single variable to a particular shock can still be non-invertible; and (2) Although our model implies a non-invertible process for output, a VAR estimation using model generated data can reproduce the IRF close to the true IRF generated by the model.

First notice that, even though the VAR procedure requires that the variables be jointly invertible with respect to the identified set of shocks, a single variable may not be invertible with respect to a particular shock. In the case of Angeletos, Collard, and Dellas (2020), though the system is invertible, this does not imply that the conditional response of output to the main business shock has to be invertible. As an example, consider the following auto-regressive bivariate system

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} = \frac{\begin{bmatrix} 1-\rho_{22}L & \rho_{12}L \\ \rho_{21}L & 1-\rho_{11}L \end{bmatrix} \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}}{(1-\rho_{11}L)(1-\rho_{22}L) - \rho_{12}\rho_{21}L^2}.$$

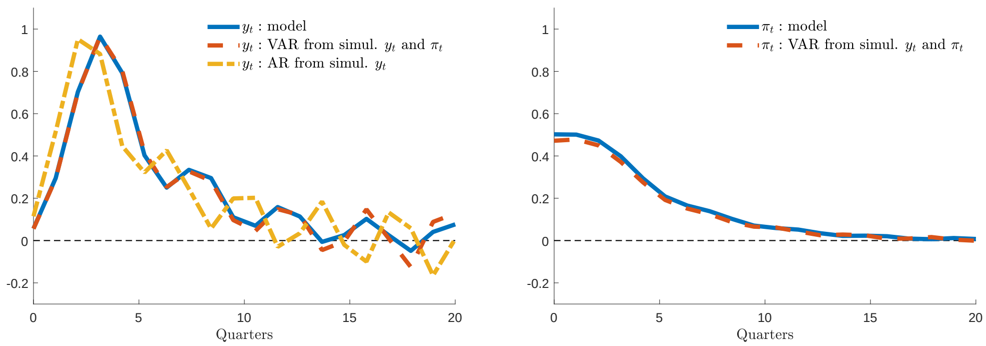
For invertility of the system we need the roots of the polynomial

$$(1-\rho_{11}L)(1-\rho_{22}L) - \rho_{12}\rho_{21}L^2,$$

to be outside the unit circle, while for invertibility of the conditional response of y_{1t} to ϵ_{1t} we need $|\rho_{22}| < 1$. It is easy to find parameters such that the bivariate system is invertible (i.e., $\{\epsilon_{1\tau}, \epsilon_{2\tau}\}_{-\infty}^{t}$ can be recovered from $\{y_{1\tau}, y_{2\tau}\}_{-\infty}^{t}$), while y_{1t} is not invertible with respect to ϵ_{1t} (i.e., $\{\epsilon_{1\tau}\}_{-\infty}^{t}$ cannot be recovered from $\{y_{1\tau}\}_{-\infty}^{t}$). The following matrix of parameters is one example:

$$\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ -2/3 & 3/2 \end{bmatrix}.$$

The main business cycle shock in Angeletos, Collard, and Dellas (2020) is identified by maximizing the volatility of the unemployment rate at a certain frequency. The identification strategy is not directly associated with an interpretable shock in a micro-founded model such as a TFP shock or a monetary policy shock. Also, the identification strategy does not impose any assumption on agents' information sets. Therefore, it could be the case that the system is invertible from the perspective of an econometrician who observed all the endogenous variables perfectly, while agents in the model only receive imperfect information as in our model environment.



(a) response of output

(b) response of inflation

Figure 3: Recovering Impulse Response Functions from Simulated Data

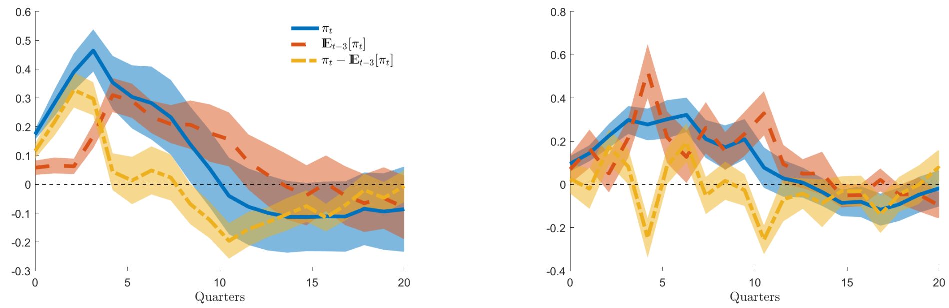
Parameters: $\rho = 0.9$, $\alpha = 0.5$, and $\tau = 0.25$.

In principle, however, it remains unclear to what extent our structural IRF can be recovered from a VAR. To address this question, we conduct two tests: in the first test, we run the same VAR exercise as in Angeletos, Collard, and Dellas (2020) on simulated data for output and inflation from our model, and compare the IRFs with the structural ones. As shown in Figure 3 above, the true IRF (solid blue line) and the response from the bivariate var (dashed red line) are quite similar to each other both quantitatively and qualitatively, though the estimated system is invertible by construction. In the second test, we estimate a univariate dynamic system with only output data generated from our model. The recovered IRF is displayed in the yellow broken line in Panel 3a, which is also not far from the structural IRF. This suggests that the magnitude of the non-invertibility in our calibrated model does not significantly affect the recovery of the IRFs from a VAR.

E.2 Quantitative Response of Inflation to a Supply shock

In Section 4.3, we compare the predictions of our model to data on unemployment expectations. Here we show that we obtain analogous results when we consider the response of the inflation and inflation forecasts to a supply shock. In the data we consider the shock orthogonal to the main business cycle shock (Angeletos, Collard, and Dellas, 2020), which drives most of the fluctuation in inflation. Figure 4 presents impulse responses computed by Angeletos, Huo, and Sastry (2020) using data from the Survey of Professional Forecasts and from University of Michigan Survey of Consumers.

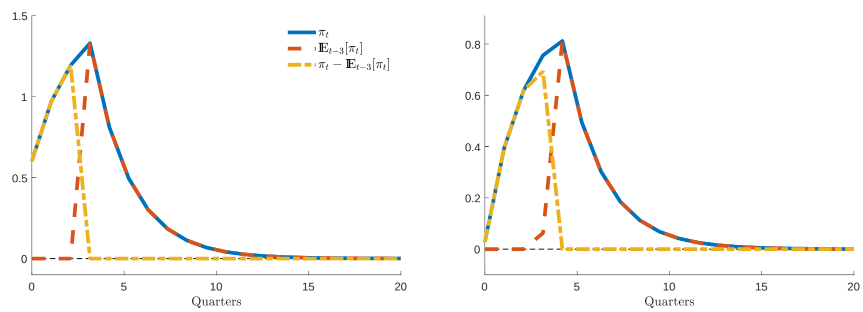
Relative to the system (13)-(16), here we additionally assume that consumers are not subject to informational frictions, so that a standard IS curve operates: $c_t = -\varsigma \mathbb{E}_t[\phi_\pi \pi_t - \pi_{t+1}] + \mathbb{E}_t[c_{t+1}]$. Figure 5 presents the results for the model under the same four information structures considered in Section 4.3. We set the economic parameters to the same level, and choose the remaining three parameters so that the impulse response of the model with endogenous information approximates the data from the SPF—the parameters are reported under



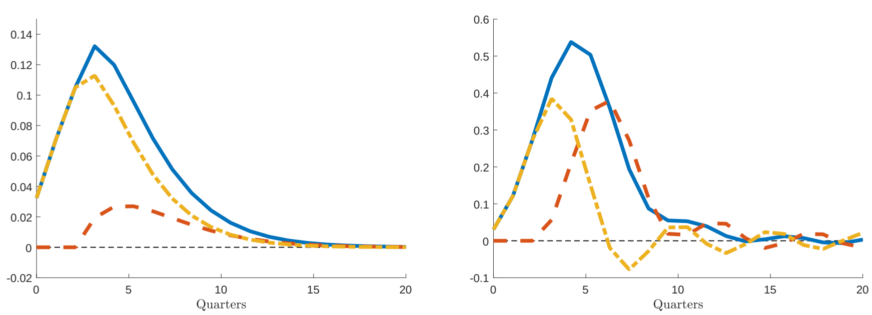
(a) inflation SPF (b) inflation Michigan

Figure 4: Responses of Annualized Inflation to Supply Shocks

the figure. In the data, observations occur at a quarterly rate but inflation rates are annualized. In the model, annualized inflation in period t is equal the sum of current quarterly rate with the preceding three rates. This explains the initial increasing pattern of the impulse response of the perfect information model, for instance. Aside from this, all qualitative features noted in Section 4.3 are also observed here.



(a) perfect information (b) perfect revelation of past



(c) exogenous information (d) endogenous information

Figure 5: Quantitative Model (Annualized) Inflation

Preset parameters: $\kappa = 0.05$, $\theta = 0.45$, $\delta = 0.99$, $\varsigma = 1$, $\mathtt{mpc} = 0.15$, and $\phi_\pi = 1.5$. Calibrated parameters: $\rho = 0.61$, $\tau = 0.09$, and $\sigma_\eta = 14.09$.

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