Infinite Higher Order Beliefs and First Order Beliefs: An Equivalence Result

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Introduction

- A large literature explores the role of dispersed information
- A common feature of these models: infinite higher order beliefs
  - important in accounting for observed dynamics in the data
  - but make it hard to characterize the equilibrium
- This paper: a novel characterization of widely used beauty contest models
  - provides new insights for some classical problems
  - allows for an easy way to solve for the equilibrium
This Paper: an Equivalence Result

- Rational expectations equilibrium (REE)

\[ y_{it} = (1 - \alpha)E_{it}[\theta_t] + \alpha E_{it}[y_t] \]
\[ y_t = \int y_{jt} \]

  - agents care about both exogenous fundamentals and others' endogenous action
  - fixed point problem with infinite higher order beliefs

- Simple forecasting problem (SFP)

\[ \tilde{y}_{it} = \tilde{E}_{it}[\theta_t] \]

  - agents only care about exogenous fundamentals
  - only first order beliefs

- For REE, there exists an "equivalent" SFP with modified information \( \tilde{E}_{it}[\cdot] \)
**Main Results**

1. $\tilde{E}_{it}[\cdot]$: expectation conditional on $\alpha$-modified signal process $\tilde{x}_{it}$
   - precisions of all idiosyncratic shocks in original signals $x_{it}$ discounted by $\alpha$

2. Equivalence result at individual level

   $$ y_{it} = (1 - \alpha)E_{it}[\theta_t] + \alpha E_{it}[y_t] = h(L)x_{it} $$

   $$ \tilde{E}_{it}[\theta_t] = h(L)\tilde{x}_{it} $$

   - the policy rule in equilibrium is the same as a simple forecasting problem

3. Equivalence result at aggregate level

   $$ \int y_{it} = (1 - \alpha) \sum_{k=0}^{\infty} \alpha^k \tilde{E}_{t+1}^{k+1}[\theta_t] = \tilde{E}_t[\theta_t] $$

   - weighted sum of infinite higher order beliefs is the same as a first order belief
Implications of Main Results

- Private signals are discounted in policy rule
  - public signals help to coordinate, in addition to forecasting fundamentals

- Dynamics of higher order beliefs v.s. first order beliefs
  - higher order beliefs are very different from first order beliefs
  - weighted sum of infinite higher order beliefs are similar to first order beliefs

- Models with complicated dynamics can be solved and quantified
  - apply this result to Woodford (2003) to identify REE v.s. SFP
Related Literature

- Static beauty contest type models
  - Morris and Shin (2002), Angeletos and Pavan (2007), and many others

- Solving models with dynamics of higher order beliefs
  2. Time domain with truncation: Hellwig and Venkateswaran (2009), Nimark (2011)
  4. Time domain without truncation: this paper
Outline

1. General setup and main results
   - static case

2. Applications with dynamic information
   - inertia: long-lasting effect
   - oscillation: overreaction and underreaction

3. Extensions of basic equivalence results
   - beauty contest models with more general best response
   - beauty contest models with multiple actions
   - beauty contest models in a network (joint with Laura Veldkamp)
A Beauty Contest Model: Best Response Function

- A continuum of agents index by $i \in [0, 1]$

  $$\min_{y_{it}} E_{it}[(1 - \alpha)(y_{it} - \theta_t)^2 + \alpha(y_{it} - y_t)^2]$$

  - $\theta_t$: payoff relevant economic fundamental
  - aggregate action
    $$y_t = \int y_{it}$$
  - $\alpha$: controls strategic complements or substitutes

- Best response function

  $$y_{it} = (1 - \alpha)E_{it}[\theta_t] + \alpha E_{it}[y_t]$$
A Beauty Contest Model: Best Response Function

- This type of best response function is common in macroeconomics

- Business cycle fluctuations, Benhabib et al. (2015), Angeletos and La’O (2009)

\[ q_{it} = (1 - \alpha)E_{it}[z_{it}] + \alpha E_{it}[q_t] \]


\[ p_{it} = (1 - \alpha)E_{it}[m_{it}] + \alpha E_{it}[p_t] \]

- Investment decision, Angeletos and Pavan (2007)

\[ k_{it} = (1 - \alpha)E_{it}[s_{it}] + \alpha E_{it}[k_t] \]
Agents receive signals $x_{it}$ every period

$x_{it}$ can be any multivariate stationary process driven by Gaussian shocks

$$x_{it} = M(L) \begin{bmatrix} \epsilon_t \\ u_{it} \end{bmatrix}, \quad \begin{bmatrix} \epsilon_t \\ u_{it} \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} \Sigma_{\epsilon} & 0 \\ 0 & \Sigma_{u} \end{bmatrix} \right)$$

- $\epsilon_t$: aggregate shocks, affect all agents’ signal
- $u_{it}$: idiosyncratic shocks, only affect agent $i$’s signal
- w.l.o.g, both $\Sigma_{\epsilon}$ and $\Sigma_{u}$ are diagonal matrices

Fundamental driven by aggregate shocks, will be relaxed later

$$\theta_t = \phi(L)\epsilon_t$$
Infinitie Higher Order Beliefs

\[ y_{it} = (1 - \alpha)E_{it}[\theta_t] + \alpha E_{it}[y_t] \]
Infinite Higher Order Beliefs

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\[ = (1 - \alpha)E_{it}[\theta_t] + \alpha E_{it} \left[ \int y_{jt} \right] \]

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\[ = (1 - \alpha)E_{it}[\theta_t] + \alpha (1 - \alpha)E_{it} \left[ \int E_{jt}[\theta_t] \right] + \alpha^2 E_{it} \left[ \int E_{jt}[y_t] \right] \]
Infinite Higher Order Beliefs

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\[ = (1 - \alpha)E_{it}[\theta_t] + \alpha(1 - \alpha)E_{it} \left[ \int E_{jt}[\theta_t] \right] + \alpha^2 (1 - \alpha)E_{it} \left[ \int E_{jt} \left[ \int E_{kt}[\theta_t] \right] \right] \]
\[ + \alpha^3 E_{it} \left[ \int E_{jt} \left[ \int E_{kt}[y_t] \right] \right] \]
\[ : \]

- Keynes (1936)’s beauty contest:

“\textit{It is not a case of choosing those that, to the best of one’s judgment, are really the prettiest, nor even those that average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practice the fourth, fifth and higher degrees.}”
Infinite Higher Order Beliefs

- Define higher order beliefs recursively
  - 1st order belief: \( E_i t [E_t^0 [\theta_t]] \) where \( E_t^0 [\theta_t] \equiv \theta_t \)
  - 2nd order belief: \( E_i t [E_t^1 [\theta_t]] \) where \( E_t^1 [\theta_t] \equiv \int E_j t [\theta_t] \)
  - 3rd order belief: \( E_i t [E_t^2 [\theta_t]] \) where \( E_t^2 [\theta_t] \equiv \int E_j t [\int E_j t [\theta_t]] \)
  - ... 
  - k-th order belief: \( E_i t [E_t^{k-1} [\theta_t]] \) where \( E_t^k [\theta_t] \equiv \int E_j t [\int E_j t [E_t^{k-1} [\theta_t]]] \)

- In equilibrium, it has to be that (given \(|\alpha| < 1\))

\[
y_{it} = (1 - \alpha) \sum_{k=0}^{\infty} \alpha^k E_i t [E_t^k [\theta_t]]
\]
Infinite Regress Problem

- $E_{it}[	heta_t]$: weighted average of current signals, and prior mean $E_{it-1}[	heta_{t-1}]$

  $$E_{it}[	heta_t] = \lambda_{11} E_{it-1}[	heta_{t-1}] + g_1 x_{it}$$

- $E_{it}E_{t}^1[	heta_t]$: current signals, and prior mean $E_{it-1}[	heta_{t-1}]$, $E_{it-1}E_{t-1}^1[	heta_{t-1}]$

  $$E_{it}^2[	heta_t] = \lambda_{21} E_{it-1}E_{t-1}^1[	heta_{t-1}] + \lambda_{22} E_{it-1}[	heta_{t-1}] + g_2 x_{it}$$

- $E_{it}E_{t}^2[	heta_t]$: current signals, and additional prior mean $E_{it-1}E_{t-1}^2[	heta_{t-1}]$

- Forecasting infinite higher order beliefs requires infinite state variables

- Equilibrium policy rule depends on all past signals
Equilibrium

- All higher order beliefs are linear combinations of current and past signals

**Definition**

Given an information process $x_{it}$, the rational expectations equilibrium is a policy rule $h(L) = \sum_{k=0}^{\infty} h_k L^k$, such that

$$y_{it} = h(L)x_{it} = \sum_{k=1}^{\infty} h_k x_{it-k} = (1 - \alpha)\mathbb{E}_{it}[\theta_t] + \alpha\mathbb{E}_{it}[y_t]$$

and

$$y_t = \int y_{it} = \int h(L)x_{it}$$

**Proposition**

If $\alpha \in (-1, 1)$, then there exists a unique rational expectations equilibrium.
Equivalence Result
\(\alpha\)-modified Information Process

- Rational Expectations Equilibrium: \(y_{it} = (1 - \alpha)\mathbb{E}_{it}[\theta_t] + \alpha\mathbb{E}_{it}[y_t]\)

\[
x_{it} = M(L) \begin{bmatrix} \epsilon_t \\ u_{it} \end{bmatrix}, \quad \begin{bmatrix} \epsilon_t \\ u_{it} \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} \Sigma_{\epsilon} & 0 \\ 0 & \Sigma_u \end{bmatrix} \right)
\]

- Simple Forecasting Problem: \(\tilde{y}_{it} = \tilde{\mathbb{E}}_{it}[\theta_t]\)

\[
\tilde{x}_{it} = M(L) \begin{bmatrix} \epsilon_t \\ \tilde{u}_{it} \end{bmatrix}, \quad \begin{bmatrix} \epsilon_t \\ \tilde{u}_{it} \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} \Sigma_{\epsilon} & 0 \\ 0 & \frac{1}{1-\alpha} \Sigma_u \end{bmatrix} \right)
\]

- Information transformation \(\tilde{x}_{it}\): \(\alpha\)-modified signal process
  - precisions of idiosyncratic shocks are discounted by \(\alpha\)
Equivalence Result for Individual Actions

**Theorem**

Given a signal process $x_{it}$, the equilibrium policy rule in REE is the same as in the SFP with $\alpha$-modified signal process $\tilde{x}_{it}$:

$$y_{it} = (1 - \alpha) \mathbb{E}_{it}[\theta_t] + \alpha \mathbb{E}_{it}[y_t] = h(L)x_{it},$$

$$\tilde{\mathbb{E}}_{it}[\theta_t] = h(L)\tilde{x}_{it}.$$

**Corollary**

The forecasting rule of a weighted sum of infinite higher order beliefs is the same as the first order belief with $\alpha$-modified signal:

$$(1 - \alpha) \sum_{k=0}^{\infty} \alpha^k \mathbb{E}_{it} \left[ \bar{\mathbb{E}}_{t}^{k}[\theta_t] \right] = h(L)x_{it}$$

$$\tilde{\mathbb{E}}_{it}[\theta_t] = h(L)\tilde{x}_{it}.$$
Equivalence Result for Aggregate Actions

**Theorem**

Given a signal process $x_{it}$, the aggregate action in REE is the same as in the SFP with $\alpha$-modified signal process $\tilde{x}_{it}$:

$$y_t = \int \tilde{E}_{it} [\theta_t] = \int h(L)\tilde{x}_{it} = h(L)M(L) \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}$$

**Corollary**

The forecasting rule of weighted sum of infinite higher order beliefs is the same as first order belief with $\alpha$-modified signal:

$$(1 - \alpha) \sum_{k=0}^{\infty} \alpha^k \tilde{E}_{t}^{k+1} [\theta_t] = \tilde{E}_{t} [\theta_t]$$
Remarks

1. We prove the equivalence result via frequency domain method, which is not necessary when applying the equivalence result.

2. When $\theta_t$ and $x_{it}$ permit a finite state representation
   - the policy rule $h(L)$ can be obtained via Kalman filter
   - the equilibrium also has a finite state representation

3. When $\theta_t$ or $x_{it}$ does not permit a finite state representation, the equivalence result still holds
   - relevant for endogenous information

4. Conjecture: equivalence result also holds for non-stationary process
Static Case
Introduction Model Main Results Applications Extensions

Static Case: Morris and Shin (2002)

- Assume $\theta$ is i.i.d and agents have improper prior about it

- Agents receive a public and a private signal about $\theta$

\[ z = \theta + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \tau_{\epsilon}^{-1}) \]

\[ x_i = \theta + u_i, \quad u_i \sim \mathcal{N}(0, \tau_{u}^{-1}) \]

- The forecast about $\theta$ is

\[ \mathbb{E}_i[\theta] = \frac{\tau_{\epsilon}}{\tau_{\epsilon} + \tau_{u}} z + \frac{\tau_{u}}{\tau_{\epsilon} + \tau_{u}} x_i \]
Static Case: Equilibrium

- Equilibrium policy rule takes the following form

\[ y_i = (1 - \alpha)E_i[\theta] + \alpha E_i[y] = h_1 z + h_2 x_i \]

- By method of undetermined coefficients

\[ y_i = \frac{\tau_\epsilon}{\tau_\epsilon + (1 - \alpha)\tau_u} z + \frac{(1 - \alpha)\tau_u}{\tau_\epsilon + (1 - \alpha)\tau_u} x_i \]

- Recall that the forecast about \( \theta_t \) is

\[ E_i[\theta] = \frac{\tau_\epsilon}{\tau_\epsilon + \tau_u} z + \frac{\tau_u}{\tau_\epsilon + \tau_u} x_i \]

- The private signals are discounted by \( \alpha \) in equilibrium
Where is the Discounting Coming From?

\[ y_i = (1 - \alpha) \sum_{k=0}^{\infty} \alpha^k \mathbb{E}_i \mathbb{E}^k [\theta] \]

\[ \mathbb{E}_i [\theta] = \frac{\tau_e}{\tau_e + \tau_u} z + \frac{\tau_u}{\tau_e + \tau_u} x_i \]

\[ \int \mathbb{E}_i [\theta] = \mathbb{E}^1 [\theta] = \frac{\tau_e}{\tau_e + \tau_u} z + \frac{\tau_u}{\tau_e + \tau_u} \theta \]

\[ \mathbb{E}_i \mathbb{E}^1 [\theta] = \frac{\tau_e}{\tau_e + \tau_u} z + \frac{\tau_u}{\tau_e + \tau_u} \left( \frac{\tau_e}{\tau_e + \tau_u} z + \frac{\tau_u}{\tau_e + \tau_u} x_i \right) \]

\[ \vdots \]

\[ \mathbb{E}_i \mathbb{E}^k [\theta] = \left[ 1 - \left( \frac{\tau_u}{\tau_e + \tau_u} \right)^{k+1} \right] z + \left( \frac{\tau_u}{\tau_e + \tau_u} \right)^{k+1} x_i \]

- As the order of higher order beliefs goes up, private signals are less important.
- Public signals are more useful in forecasting the aggregate action.
REE and SFP in Static Case

- Rational Expectations Equilibrium: \( y_i = (1 - \alpha)E_i[\theta] + \alpha E_i[y] \)
  - signals: \( z = \theta + \epsilon, \quad \epsilon \sim N(0, \tau^{-1}_\epsilon) \)
  - \( x_i = \theta + u_i \quad u_i \sim N(0, \tau^{-1}_u) \)

- Simple Forecasting Problem: \( \tilde{y}_i = \tilde{E}_i[\theta] \)
  - signals: \( z = \theta + \epsilon, \quad \epsilon \sim N(0, \tau^{-1}_\epsilon) \)
  - \( \tilde{x}_i = \theta + \tilde{u}_i \quad \tilde{u}_{it} \sim N(0, ((1 - \alpha)\tau_u)^{-1}) \)

- \( \alpha \)-modified signal: precision of private shock is discounted by \( \alpha \)
Apply Equivalence Results in Static Model

- Individual policy rule is the same in REE and SFP

\[ y_i = \frac{\tau_e}{\tau_e + (1 - \alpha)\tau_u} z + \frac{(1 - \alpha)\tau_u}{\tau_e + (1 - \alpha)\tau_u} x_i \]

\[ \tilde{E}_i[\theta] = \frac{\tau_e}{\tau_e + (1 - \alpha)\tau_u} z + \frac{(1 - \alpha)\tau_u}{\tau_e + (1 - \alpha)\tau_u} \tilde{x}_i \]

- Aggregate allocation is the same in REE and SFP

\[ y = \int y_i = \int \tilde{E}_i[\theta] = \frac{\tau_e}{\tau_e + (1 - \alpha)\tau_u} z + \frac{(1 - \alpha)\tau_u}{\tau_e + (1 - \alpha)\tau_u} \theta \]

- Sum of infinite higher order beliefs is the same as first order belief

\[ (1 - \alpha) \sum_{k=0}^{\infty} \alpha^k \tilde{E}^{k+1}[\theta] = \tilde{E}[\theta] \]
Applications
Application I: Inertia

- Inertia: hump-shaped response

- Noisy business cycles Angeletos and La’O (2009):
  - Heterogeneity of information can contribute to significant inertia in the response of macroeconomic outcomes to such shocks

- Effects of monetary policy shock Woodford (2003):
  - Higher-order expectations adjust only sluggishly to a disturbance, even when the public’s average estimate of what has occurred adjusts fairly rapidly. If strategic complementarities in price-setting are strong enough, the real effects of a nominal disturbance may be both large and highly persistent.
Example: AR(1) Signal

- The same best response function

\[ y_{it} = (1 - \alpha)E_{it}[\theta_t] + \alpha E_{it}[y_t] \]

- Assume the fundamental \( \theta_t \) follows an AR(1) process

\[ \theta_t = \rho \theta_{t-1} + \epsilon_t \quad \epsilon_t \sim \mathcal{N}(0, \tau_{\epsilon}^{-1}) \]

- Agents receive a private signal about \( \theta_t \)

\[ x_{it} = \theta_t + u_{it} \quad u_{it} \sim \mathcal{N}(0, \tau_{u}^{-1}) \]
Understanding Inertia via Infinite Higher Order Beliefs

- Recall that agent $i$’s best response is a weighted average of all higher order beliefs

\[ y_{it} = (1 - \alpha) \sum_{k=0}^{\infty} \alpha^k E_{it} \left[ \mathbb{E}^k_t [\theta_t] \right] \]

- Higher order beliefs display inertia, so does the aggregate action
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- Higher order beliefs display inertia, so does the aggregate action
Apply Equivalence Result

• In the AR(1) example, the corresponding $\alpha$-modified signal is

$$\tilde{x}_{it} = \theta_t + \tilde{u}_{it} \quad \tilde{u}_{it} \sim \mathcal{N}(0, ((1 - \alpha)\tau_u)^{-1})$$

• Solution to the corresponding simple forecasting problem

$$\mathbb{E}_{it}[\theta_t] = \frac{\kappa}{\kappa + (1 - \alpha)\tau_u} \rho \mathbb{E}_{it-1}[\theta_{t-1}] + \frac{(1 - \alpha)\tau_u}{\kappa + (1 - \alpha)\tau_u} \tilde{x}_{it}$$

$$\kappa = \left[ \rho^2 (\kappa + (1 - \alpha)\tau_u)^{-1} + \tau_\epsilon^{-1} \right]^{-1}$$

• The law of motion in rational expectations equilibrium

$$y_{it} = \frac{\kappa}{\kappa + (1 - \alpha)\tau_u} \rho y_{it-1} + \frac{(1 - \alpha)\tau_u}{\kappa + (1 - \alpha)\tau_u} x_{it}$$
Apply Equivalence Result

- In the AR(1) example, the corresponding $\alpha$-modified signal is

$$\tilde{x}_{it} = \theta_t + \tilde{u}_{it} \quad \tilde{u}_{it} \sim \mathcal{N}(0, ((1 - \alpha)\tau_u)^{-1})$$

- Solution to the corresponding simple forecasting problem

$$\tilde{E}_{it}[\theta_t] = \frac{\kappa}{\kappa + (1 - \alpha)\tau_u} \rho \tilde{E}_{it-1}[\theta_{t-1}] + \frac{(1 - \alpha)\tau_u}{\kappa + (1 - \alpha)\tau_u} \tilde{x}_{it}$$

$\phi$: Kalman gain

$$\kappa = \left[ \rho^2 (\kappa + (1 - \alpha)\tau_u)^{-1} + \tau_\epsilon^{-1} \right]^{-1}$$

- The law of motion in rational expectations equilibrium

$$y_{it} = \phi \rho y_{it-1} + (1 - \phi) x_{it}$$

$$y_t = \phi \rho y_{t-1} + (1 - \phi) \theta_t$$
Understanding the Inertia via the Equivalence Result

- Impulse response of aggregation to a shock to $\theta_t$ at time 0

$$y_t - y_{t-1} = \rho^{t-1} \left[ \phi^t (1 - \phi \rho) - (1 - \rho) \right], \quad \text{for all } t \geq 0$$

where

$$\phi = \frac{\kappa}{\kappa + (1 - \alpha) \tau_u}$$

- Necessary and sufficient condition for a hump-shaped response

$$\phi > \frac{1}{\rho} - 1$$

- small $\tau_u$: private signals not very precise
- large $\alpha$: high strategic complementarity, discount the signal
Application II: Oscillation

- Oscillation: overreaction and underreaction to fundamentals

- Waves of optimisms and pessimisms, Rodina and Walker (2012)

- Excess volatility of prices and returns, Kasa, et.al (2014)
Example: MA(1) Signal

- The same best response function

\[ y_{it} = (1 - \alpha)E_{it}[\epsilon_t] + \alpha E_{it}[y_t] \]

- Assume the fundamental \( \epsilon_t \) follows an i.i.d process, \( \epsilon_t \sim \mathcal{N}(0, 1) \)

- Agents receive a private signal about \( \epsilon_t \)

\[ x_{it} = \epsilon_t + \lambda \epsilon_{t-1} + u_{it} \quad u_{it} \sim \mathcal{N}(0, \tau_u^{-1}) \]

where \( \lambda \in (0, 1) \)
Infinite Higher Order Beliefs

- Recall that agent $i$’s best response is a weighted average of all higher order beliefs:

$$y_{it} = (1 - \alpha) \sum_{k=0}^{\infty} \alpha^k \mathbb{E}_{it} \left[ \mathbb{E}_t^k \left( \epsilon_t \right) \right]$$

- Signal: $x_{it} = \epsilon_t + \lambda \epsilon_{t-1} + u_{it}$
Recall that agent $i$’s best response is a weighted average of all higher order beliefs

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Signal: $x_{it} = \epsilon_t + \lambda \epsilon_{t-1} + u_{it}$
Apply Equivalence Result

- In the MA(1) example, the corresponding $\alpha$-modified signal is

$$\tilde{x}_{it} = \epsilon_t + \lambda \epsilon_{t-1} + \tilde{u}_{it} \quad \tilde{u}_{it} \sim \mathcal{N}(0, ((1 - \alpha)\tau_u)^{-1})$$

- Solution to the corresponding simple forecasting problem

$$\E_{it}[\epsilon_t] = \frac{\vartheta}{\lambda(1 + \vartheta L)} \tilde{x}_{it}$$

$$\vartheta = \frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda(1 - \alpha)\tau_u} - \sqrt{\left( \frac{1}{\lambda} + \frac{1}{\lambda(1 - \alpha)\tau_u} \right)^2 - 4} \right)$$

- The law of motion in rational expectations equilibrium

$$y_{it} = \frac{\vartheta}{\lambda(1 + \vartheta L)} x_{it}$$

$$y_{t} = \frac{\vartheta}{\lambda(1 + \vartheta L)} (\epsilon_t + \lambda \epsilon_{t-1})$$
Understanding the Oscillation Behavior

- Signal: \( \tilde{x}_{it} = \epsilon_t + \lambda \epsilon_{t-1} + \tilde{u}_{it} \)

- The impulse response of \( \tilde{E}_{it}[\epsilon_t] \) to \( \epsilon_t \) shock

\[
\tilde{E}_{it}[\epsilon_t] = \begin{cases} 
\frac{\vartheta}{\lambda}, & \text{if } t = 0 \\
-\frac{1}{\lambda} (\lambda - \vartheta) (-\vartheta)^t, & \text{if } t > 0
\end{cases}
\]

- Underestimate \( \epsilon_t \), then overestimate \( \epsilon_{t+1} \)
  
  - underestimate \( \epsilon_0 \): \( \tilde{E}_{i0}[\epsilon_0] = \frac{\vartheta}{\lambda} < \epsilon_0 = 1 \)
  
  - overestimate \( \epsilon_1 \): \( \tilde{E}_{i1}[\epsilon_1] = \frac{\vartheta}{\lambda} (\lambda - \vartheta) > \epsilon_1 = 0 \)
  
  - underestimate \( \epsilon_2 \): \( \tilde{E}_{i2}[\epsilon_2] = -\frac{\vartheta^2}{\lambda} (\lambda - \vartheta) < \epsilon_2 = 0 \)
Understanding the Oscillation Behavior

- Persistence of estimates $\vartheta$ monotonically increases in $(1 - \alpha)\tau_u$

- Size of undere(over)stimation does not monotonically increase in $(1 - \alpha)\tau_u$

$$\left| \epsilon_t - \tilde{E}_{it}[\epsilon_t] \right| = \begin{cases} 
1 - \frac{\vartheta}{\lambda}, & \text{if } t = 0 \\
\frac{1}{\lambda} (\lambda - \vartheta) \vartheta^t, & \text{if } t > 0
\end{cases}$$

- if $(1 - \alpha)\tau_u$ high, $\epsilon_t$ can be accurately estimated, little undere(over)stimation

- if $(1 - \alpha)\tau_u$ low, most variations of $x_{it}$ are attributed to noise $u_{it}$
Understanding the Oscillation Behavior

- Magnitude of oscillation does not change monotonically with $\alpha$
Oscillation with Persistent Fundamental

Suppose agents best response is

\[ y_{it} = (1 - \alpha)E_{it}[\theta_t] + \alpha E_{it}[y_t] \]

where

\[ \theta_t = \rho \theta_{t-1} + \epsilon_t \]

- VAR evidence on the effects of monetary policy shock
  - hump-shaped response of output and inflation
  - inertia in dynamics

- Woodford (2003): inertia can be the result of dispersed information
  - inflation and output are driven by all higher order beliefs
  - higher order beliefs are more persistent, harder to change

- Can we tell whether the inertia is due to higher order or first order beliefs?
Setup in Woodford (2003)

- Individual firm $i$’s pricing decision

\[ p_{it} = (1 - \alpha) \mathbb{E}_{it}[q_t] + \alpha \mathbb{E}_{it}[p_t] \]

- Aggregate variables
  
  - $q_t$: exogenous aggregate nominal expenditure
  
  - inflation rate: $\pi_t = p_t - p_{t-1}$
  
  - real output: $y_t = q_t - p_t$

- Signals
  
  - growth rate of nominal expenditure is driven by monetary shock $\epsilon_t$

\[ \Delta q_t = \rho \Delta q_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \tau_\epsilon^{-1}) \]

  - firms observe a private signal

\[ x_{it} = q_t + u_{it}, \quad u_{it} \sim \mathcal{N}(0, \tau_u^{-1}) \]
Apply Equivalence Result

- The corresponding $\alpha$-modified signal is

$$\tilde{x}_{it} = q_t + \tilde{u}_{it} \quad \tilde{u}_{it} \sim \mathcal{N}(0, (1 - \alpha) \tau_u)^{-1}$$

- Solution to the corresponding simple forecasting problem

$$\tilde{E}_{it}[\theta_t] = \frac{\phi - \kappa \left(\phi + \rho \kappa\right) \left(\phi - \rho \left(1 + \rho\right) \kappa L\right)}{(\phi + \rho \kappa) \phi^2 - (1 + \rho)^2 \kappa \phi^2 L + \rho \left(1 + \rho\right) \left(\rho^2 \left(\kappa - \phi\right) \kappa^2 + \rho \kappa^3 + \kappa \phi^2\right) L^2} \tilde{x}_{it}$$

$$\phi = \kappa + (1 - \alpha) \tau_u$$

$$\kappa = \left(\frac{(1 + \rho)^2 \phi^2 + \rho^2 \left(\phi - \rho^2 \kappa\right) \left(\phi - \kappa\right)}{\phi \left(\phi + \rho \kappa\right)^2} + \tau_e^{-1}\right)^{-1}$$

- The policy rule in rational expectations equilibrium

$$p_{it} = \frac{\phi - \kappa \left(\phi + \rho \kappa\right) \left(\phi - \rho \left(1 + \rho\right) \kappa L\right)}{(\phi + \rho \kappa) \phi^2 - (1 + \rho)^2 \kappa \phi^2 L + \rho \left(1 + \rho\right) \left(\rho^2 \left(\kappa - \phi\right) \kappa^2 + \rho \kappa^3 + \kappa \phi^2\right) L^2} x_{it}$$
Impulse Response of Inflation and Output

- Define $\tilde{\tau}_u = (1 - \alpha)\tau_u$

- As $\tilde{\tau}_u$ decreases, the peak of impulse response delays

![Graphs showing impulse response of inflation and output with varying $\tilde{\tau}_u$.]
Can the Model Match the Impulse Response in the Data?

- Choose $\rho, \widetilde{\tau}_u, \tau_\epsilon$ to match IRF of inflation and output in the first 15 periods
- With relatively small $\widetilde{\tau}_u$, we can match the hump-shaped response

![Graphs showing inflation and output impulse responses](image)

By equivalence result, REE with $\tau_u = \frac{\widetilde{\tau}_u}{1-\alpha}$ yields the same impulse response

Source: Christiano et al. (2005)
Identifying REE vs SPF using Survey Data

- To obtain the same aggregate allocation
  - REE displays smaller forecast dispersion and forecast error
- Data on individual forecast errors help to distinguish REE from SFP

REE with $\alpha = 0.95$ can match cross-sectional belief dispersion in the Survey of Professional Forecasters
Extensions
Extension I: Arbitrary Fundamentals in Best Response Function

- So far, the equivalence result works for

\[ y_{it} = (1 - \alpha)E_{it}[\theta_t] + \alpha E_{it}[y_t] \]

- Action may depend on any shock

\[ y_{it} = \gamma E_{it}[\xi_{it+k}] + \alpha E_{it}[y_t] \]

  - \( \xi_{it+k} \) can depend on aggregate or idiosyncratic shocks
  - \( \xi_{it+k} \) can depend on future or past shocks

- A variation of the equivalence results applies for this case
Extension I: Arbitrary Fundamentals in Best Response Function

- By Kalman filter, $\mathbb{E}_{it}[\xi_{it+k}] = g_1(L)e_t + g_2(L)u_{it} = \varphi_t + \omega_{it}$

- In equilibrium

$$y_{it} = \gamma(\varphi_t + \omega_{it}) + \alpha \mathbb{E}_{it}[y_t]$$

$$= \gamma(\varphi_t + \omega_{it}) + \alpha \gamma \sum_{k=0}^{\infty} \alpha^k \mathbb{E}_{it} \left[ \mathbb{E}_{t}^k [\varphi_t] \right]$$

- Recall we have shown that

$$(1 - \alpha) \sum_{k=0}^{\infty} \alpha^k \mathbb{E}_{it} \left[ \mathbb{E}_{t}^k [\varphi_t] \right] = h(L)x_{it}$$

$$\tilde{\mathbb{E}}_{it} [\varphi_t] = h(L)\tilde{x}_{it}$$

- Therefore, the individual policy rule is

$$y_{it} = \gamma(\varphi_t + \omega_{it}) + \frac{\alpha \gamma}{1 - \alpha} h(L)x_{it}$$
Extension I: Example

- Noisy business cycles Angeletos and La’O (2009):
  \[ y_{it} = \gamma(\varphi_t + \omega_{it}) + \alpha \mathbb{E}_{it}[y_t] \]
  - \( \omega_{it}, \varphi_t \): idiosyncratic and aggregate components of TFP
  - agents observe \( \varphi_t + \omega_{it} \), but do not know \( \varphi_t \) perfectly

- By our equivalence result,
  \[
  (1 - \alpha) \sum_{k=0}^{\infty} \alpha^k \mathbb{E}_{it} \left[ \mathbb{E}_{it}^k[\varphi_t] \right] = h(L)x_{it}
  \]
  \[
  \mathbb{E}_{it}[\varphi_t] = h(L)\tilde{x}_{it}
  \]

- The individual policy rule is
  \[ y_{it} = \gamma(\varphi_t + \omega_{it}) + \frac{\alpha \gamma}{1 - \alpha} h(L)x_{it} \]
Extension II: Multiple Actions

Suppose there are more than one action that an agent needs to choose

\[ y_{it} = \mathbb{E}_{it}[\theta_t] + A \mathbb{E}_{it}[y_t] \]

\[
\begin{bmatrix}
  y_{it,1} \\
  y_{it,2} \\
  \vdots \\
  y_{it,n}
\end{bmatrix} = \begin{bmatrix}
  \mathbb{E}_{it}[\theta_{t,1}] \\
  \mathbb{E}_{it}[\theta_{t,2}] \\
  \vdots \\
  \mathbb{E}_{it}[\theta_{t,n}]
\end{bmatrix} + \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix} \begin{bmatrix}
  \mathbb{E}_{it}[y_{t,1}] \\
  \mathbb{E}_{it}[y_{t,2}] \\
  \vdots \\
  \mathbb{E}_{it}[y_{t,n}]
\end{bmatrix}
\]
Extension II: Multiple Actions

• If $A$ is diagonalizable and all eigenvalues are within the unit circle

$$y_{it} = \sum_{k=0}^{\infty} A^k E_{it} [E_t [\theta_t]]$$

$$= \sum_{k=0}^{\infty} U \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix}^k U^{-1} E_{it} [E_t [\theta_t]]$$

$$= \sum_{j=1}^{n} UE_{E,j} V \sum_{k=0}^{\infty} \omega_j^k E_{i,t} [E_t [\theta_t]]$$

SFP with $\omega_j$-modified signal

• $y_{it}$ is a weighted average of different first order beliefs
Extension III: Network

- Suppose there are $n$ agents in the economy.

- Agent $i$’s best response function is

$$y_{it} = E_{it}[\theta_t] + \sum_{j=1}^{n} W_{ij} E_{it}[y_{jt}]$$

- The fundamental is a stationary process driven by common shocks

$$\theta_t = \phi(L) \eta_t$$

- Agent $i$ receives a public signal $z_t$ and a private signal $x_{it}$ about $\theta_t$

$$z_t = \theta_t + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \tau_\epsilon^{-1})$$

$$x_{it} = \theta_t + u_{it}, \quad u_{it} \sim \mathcal{N}(0, \tau_u^{-1})$$
Extension III: Equivalence Result

- $k$-th modified signal process

\[ z_t = \theta_t + \epsilon_t, \quad \epsilon_{it} \sim \mathcal{N}(0, \tau_\epsilon^{-1}) \]
\[ \tilde{x}_{it} = \theta_t + \tilde{u}_{it}, \quad u_{it} \sim \mathcal{N}(0, \tilde{\tau}_k^{-1}) \]

where $\tilde{\tau}_k^{-1}$ is the $k$-th eigenvalue of $\tau_u(I - W)^{-1}$.

- Simple forecasting problem with $k$-th modified signal process

\[ \tilde{E}_{kt}[\theta_t] = h^1_k(L)z_t + h^2_k(L)\tilde{x}_{it} \]

- Agent $i$’s policy rule is a weighted average of $n$ different forecasting problems

\[ y_{it} = \sum_{k=1}^{n} \left( \mathbf{U}_{ik} \mathbf{U}_k^{-1} \mathbf{1}_{n \times 1} \right) \left[ h^1_k(L)z_t + h^2_k(L)\tilde{x}_{it} \right] \]

where $\mathbf{U}_k$ is the $k$-th eigenvector of $\tau_u(I - W)^{-1}$. 
Conclusion

- An equivalence result between REE and SFP with modified signals
  - individual policy rule in REE is the same
  - sum of infinite higher order beliefs is the same as a first order belief

- Provide an easy solution and characterization for beauty contest models
  - help understanding inertia and oscillation
  - useful for quantitative exercise

- This equivalence result also extends to models with network