Bias and Sensitivity under Ambiguity *

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Abstract

This paper characterizes the effects of ambiguity aversion under dispersed information. The equilibrium outcome is observationally equivalent to a Bayesian forecast of the fundamental with increased sensitivity to signals and a pessimistic bias. This equivalence result takes a simple form that accommodates dynamic information and strategic interactions. Applying the result, we show that ambiguity aversion helps rationalize the joint empirical pattern between the bias and persistence of inflation forecasts conditional on household income. In a policy game à la Barro and Gordon (1983) with ambiguity-averse agents, the policy rule features higher average inflation and increased responsiveness to fundamentals.

Keywords: Ambiguity, Incomplete information, Coordination, Bias
JEL classifications: E20, E32, F44

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1 Introduction

Workhorse macroeconomic models often assume that agents have perfect knowledge about the underlying data-generating process of the economy. However, households and firms often face model uncertainty or ambiguity[1] when making economic decisions. For example, after the pandemic, inflation has surged and average inflation targeting entered the policy discussion. It is less clear whether inflation will fluctuate around 2% or 4% in the near future, and each of these scenarios could be interpreted as a different model economy. Moreover, when economic decisions are interdependent, coordination among market participants relies on their perceptions of how others would react to such uncertainty. In this environment, what are the macroeconomic effects of ambiguity? Can ambiguity help explain the deviations from rational expectations observed in survey data? How does the presence of ambiguity aversion in the private sector affect optimal policy design?

This paper makes two contributions. Theoretically, we show that despite the complexities associated with dispersed information, coordination motives, and ambiguity aversion, the equilibrium strategy is equivalent to a single-agent forecasting problem with two modifications: an amplified responsiveness to signals and a permanent pessimistic bias. This equivalence result applies to general information processes and takes a simple form, enabling the use of standard numerical algorithms to compute equilibrium strategies. On the applied side, we document that the bias in inflation forecasts decreases with household income, while the persistence of forecast errors increases with household income. This observed joint distribution can be naturally accounted for in a model where agents exhibit ambiguity aversion towards inflation shocks. In a policy game à la Barro and Gordon (1983), we show that policymakers respond to this behavioral pattern by increasing both their responsiveness to shocks and the unconditional level at which they set the inflation rate.

Framework. We consider an abstract Gaussian-quadratic economy. Agents' payoffs depend on an exogenous fundamental, their own actions, and the average actions of others, following Angeletos and Pavan (2007). The generic quadratic utility function can be regarded as the reduced form representation of a micro-founded economy that allows for general equilibrium (GE) effects. In addition, we impose minimal restrictions on the information structure to accommodate persistent learning and dispersed information. The main departure from the existing literature is that the fundamental is not only stochastic but also ambiguous, in the sense that agents do not have perfect knowledge about its objective probability distribution. In the baseline specification, agents are ambiguity-averse, with preferences represented by the smooth model of ambiguity proposed by Klibanoff, Marinacci, and Mukerji (2005). We adopt the smooth rule when updating preferences (Hanany and Klibanoff,

[1] According to Marinacci (2015), ambiguity refers to subjective uncertainty over probabilities due to a lack of ex-ante information to determine a specific model for the economy in the course of decision-making.

2009), ensuring dynamic consistency and optimal ex-ante equilibrium strategies.[2] We also extend the analysis to models with robust preferences (Hansen and Sargent, 2001a,b).

The interaction between ambiguity aversion and dispersed information makes the equilibrium difficult to characterize. It has been widely recognized that ambiguity-averse agents behave as if their beliefs about the fundamental are distorted (Ilut and Schneider, 2014; Bhandari, Borovička, and Ho, 2023) and that these distortions are affected by the equilibrium strategies. This results in a fixed-point problem between the strategies and beliefs of all agents in equilibrium. On top of that, the presence of imperfect coordination and persistent information leads to the infinite regress problem (Townsend, 1983), with higher-order beliefs potentially requiring an infinite-dimensional state space.

Equivalence results. Conceptually, the presence of ambiguity amounts to a more diffuse prior, and ambiguity aversion makes the diffusion loom larger. As a result, agents divert attention from their priors to their signals, leading to a higher sensitivity to signals. Meanwhile, in the face of model uncertainty, ambiguity-averse agents are more concerned about models that yield lower payoffs and tend to place greater emphasis on adverse probability distributions. This consideration leads agents to behave as if they are pessimistically biased when making forecasts.

Our equivalence result formalizes this intuition and circumvents the aforementioned technical complexities. The equilibrium strategy is ultimately equivalent to that of a modified single-agent Bayesian forecasting the fundamental. We first introduce a (w, α)-modified signal process. In this auxiliary forecasting problem, the precision of idiosyncratic shocks is discounted by the degree of strategic complementarity, α, which captures the idea that the coordination motive reduces the importance of private information in inferring the aggregate outcome (Huo and Pedroni, 2020). In addition, the prior variance of the fundamental shock is amplified by w—an endogenous object that summarizes the extent to which the prior becomes more diffuse due to ambiguity aversion.

We establish that the equilibrium strategy coincides with the Bayesian forecasting rule using the (w, α)-modified signal process, with two notable adjustments governed by the endogenous variable w. First, there is an additional uniform overreaction to all signals. This overreaction distinguishes the equilibrium strategy from a Bayesian forecasting problem, resembling the departure from rationality implied by diagnostic expectations (Bordalo, Gennaioli, Ma, and Shleifer, 2020). Second, there is an additional bias that is independent of the signal realization, further differentiating the equilibrium allocation from its rational-expectations counterpart. To close the loop, we provide the condition that w needs to satisfy, which involves only unconditional moments about endogenous aggregate outcomes.

[2] This approach is equivalent to characterizing the sequential equilibrium with ambiguity, an equilibrium refinement proposed in Hanany, Klibanoff, and Mukerji (2020).

This result provides a concise summary of the effects of ambiguity aversion in a general equilibrium setting. The characterization enables the derivation of general comparative static results. For example, we show that an increase in the degree of strategic complementarity leads to greater bias. To understand the underlying intuition, one needs to invoke the fact that, with ambiguity aversion, equilibrium outcomes depend on *subjective* higher-order beliefs. With rational expectations, beliefs of higher order rely more on the common prior (Morris and Shin, 2002). With ambiguity aversion, the bias in others' beliefs is embedded in the common prior, which then accumulates as the order increases. A higher degree of strategic complementarity increases the relative weight of higher-order beliefs, thereby amplifying the bias.

Computationally, this result provides a tractable algorithm for computing the equilibrium strategies. In the absence of ambiguity, dynamic higher-order expectations require, in principle, the entire history of signals as state variables. However, a finite-state representation is possible when signals follow ARMA(p, q) processes (Woodford, 2003; Angeletos and La'O, 2010; Huo and Pedroni, 2020). Our results imply that a similar finite-state equilibrium representation is still possible even with ambiguity-averse agents. The original infinite-dimensional problem collapses to a one-dimensional problem to determine the endogenous amplifier of the prior variance, w. To solve the Bayesian forecasting problem with the (w, α)-modified signal process, the standard Kalman filter can be applied.

Survey evidence on inflation expectations. Our equivalence result helps explain the patterns observed in survey data regarding the joint behavior of bias and persistence in inflation forecasts. Using the Michigan Survey of Consumers (MSC) and the Survey of Consumer Expectations (SCE), we document that the bias and the persistence of inflation forecast errors jointly vary with household income levels. Specifically, higher-income households tend to exhibit lower forecast bias but more persistent forecast errors. This joint behavior of bias and persistence is at odds with the predictions of rational expectations models. Rationality would imply an average bias of zero, and the higher persistence of forecast errors of the rich would require that they have less precise information.

We demonstrate that the observed patterns naturally arise from a micro-founded optimal consumption problem in which households face ambiguity about the inflation process. With rigid nominal incomes, higher inflation erodes households' real purchasing power. Accordingly, ambiguity-averse households assign more weight to high-inflation models, resulting in an upward bias in inflation forecasts. At the same time, ambiguity aversion induces increased sensitivity to signals or a reduction in the weight put on the prior mean of inflation, which in turn reduces the persistence of forecast errors.

Importantly, households with lower nominal incomes are more exposed to variations in inflation and are more sensitive to ambiguous inflation shocks. In the cross-section of households, it is as if higher-income households were less ambiguity-averse. Consequently, the inflation forecasts of the income-rich feature relatively lower levels of bias, lower sensitivity to signals, and higher persistence of forecast errors. With this rather parsimonious structure, our model matches the documented cross-sectional patterns reasonably well when calibrated to the data. Further, the predictions of the model are also broadly consistent with existing survey evidence on expectations, including underreaction at the consensus level (Coibion and Gorodnichenko, 2015), over-reaction at the individual level (Bordalo, Gennaioli, Ma, and Shleifer, 2020), and delayed overshooting (Angeletos, Huo, and Sastry, 2021).

Optimal policy with ambiguity aversion. How should policy respond to the observed deviations from rational expectations? To shed light on this question, we explore a policy game à la Barro and Gordon (1983). Specifically, a policymaker attempts to strike a balance between minimizing the unemployment rate and minimizing deviations from the inflation target, subject to the Phillips curve. The optimal inflation policy is then a weighted average of the exogenous random inflation target and the average inflation expectation in the private sector. Departing from Barro and Gordon (1983), we allow agents to receive dispersed information and face ambiguity about the inflation target. This ambiguity may arise from factors such as ambiguous policy communication or a loose policy objective like average inflation targeting.

The optimal policy rule is an affine function of the exogenous inflation target. The slope represents the responsiveness to changes in the target, while the intercept determines the long-run average inflation rate. In the presence of imperfect information but no ambiguity, the slope is dampened due to the underreaction of the consensus forecasts, but the intercept coincides with the mean of the inflation target. Introducing ambiguity results in additional sensitivity to information and a permanent positive bias, so a steeper slope and a lifted intercept in the policy rule. The former brings the inflation dynamics closer to the first-best, while the latter pushes the policy away from it. We show that it is always beneficial to have some ambiguity about the inflation target as the initial benefits from increased sensitivity outweigh the initial costs associated with the higher bias. However, for high enough ambiguity this relationship flips, and additional ambiguity reduces welfare. It follows that an intermediate level of ambiguity is desirable.

Robust preferences Although our analysis focuses on the smooth model of ambiguity, the main insights on bias and sensitivity extend to models with robust preferences. Despite these being significantly different approaches to ambiguity, we find that under certain conditions, for a robust preferences model, a corresponding smooth model of ambiguity exists such that the equilibrium strategies in both models are identical up to a constant. There are subtle differences in determining w for the auxiliary (w, α)-modified signal process,[3] but the quantitative predictions of the two models

[3] With robust preferences, the condition that w needs to satisfy involves conditional moments about individual variables, whereas the condition in the smooth model involves only unconditional moments of aggregate variables.

in our applications are notably similar.

Related literature This paper contributes to the literature exploring the implications of ambiguity and ambiguity aversion in macroeconomic models. There are three prominent representations of ambiguity-averse preferences in the literature: (1) the multiple priors preference axiomatized by Gilboa and Schmeidler (1989); (2) the robust preferences model proposed by Hansen and Sargent (2001a,b); and (3) the smooth model of ambiguity axiomatized by Klibanoff, Marinacci, and Mukerji (2005). Each of these representations has been extensively used in macroeconomic applications. For example, in the context of business cycle models, Ilut and Schneider (2014), Bianchi, Ilut, and Schneider (2017), and Ilut and Saijo (2021) employ the multiple priors preference approach;[4] Luo and Young (2010), Bidder and Smith (2012), and Bhandari, Borovička, and Ho (2023) use robust preferences; and Backus, Ferriere, and Zin (2015) Altug, Collard, Çakmakl, Mukerji, and Özsöylev (2020), and Pei (2023) apply the smooth model of ambiguity. In the asset pricing literature, Epstein and Wang (1994), Chen and Epstein (2002), and Miao (2009) use the multiple priors preference approach; Hansen, Sargent, and Tallarini (1999) and Anderson, Hansen, and Sargent (2003) utilize robust preferences; and Ju and Miao (2012), Collard, Mukerji, Sheppard, and Tallon (2018), and Gallant, R Jahan-Parvar, and Liu (2018) employ the smooth model of ambiguity. Additionally, Michelacci and Paciello (2019) study the effects of monetary policy announcements under multiple priors preferences.

Most of the aforementioned works assume representative agents and abstract from incomplete information. We analyze the effects of ambiguity and ambiguity aversion within a flexible environment that accommodates not only GE considerations but also incomplete information and persistent learning. In this regard, our findings complement the literature on games with incomplete information (Morris and Shin, 2002; Woodford, 2003; Angeletos and Pavan, 2007; Angeletos and La'O, 2010), extending the analysis beyond the rational expectations benchmark. Our theoretical results also establish a link between the equilibrium outcomes in the smooth model of ambiguity and the robust preferences model. In this vein, Cerreia-Vioglio, Corraob, and Lanzani (2024) characterize the effects of ambiguity in the high-coordination limit under general preferences, while our results apply for any level of coordination when restricting attention to more stylized preferences.

Our paper is also related to an extensive body of literature examining systematic biases in agents' expectations using survey data. Elliott, Komunjer, and Timmermann (2008) present evidence of systematic bias in professional forecasters' expectations and suggest that an asymmetric loss function rationalizes the documented biases.[5] Similar evidence of biased expectations has been documented

[4] We refer to Ilut and Schneider (2023) for a more comprehensive review of recent development in the applications of multiple priors preferences in macroeconomics.

[5] Pope and Schweitzer (2011) demonstrate that bias originating from loss aversion can persist even in high-stake contexts using data on the performance of professional golfers in the PGA TOUR.

by Kohlhas and Robertson (2024) and Farmer, Nakamura, and Steinsson (2023) using the Survey of Professional Forecasters. Kohlhas and Robertson (2024) show that professional forecasters' expectations are biased but more accurate than commonly used time-series models, particularly in the short run. They propose a theory of cautious expectations in which agents estimate the optimal weight on observed signals using classical inference, resulting in a trade-off between bias and accuracy. Farmer, Nakamura, and Steinsson (2023) present evidence of bias in macroeconomic expectations using the Survey of Professional Forecasters (SPF) and note that the associated forecast errors are serially correlated. Within a Bayesian paradigm that retains rationality, the authors demonstrate that slow learning over the long-run trend with a unit root can rationalize the documented bias and persistence in forecast errors. In a similar context, Andolfatto, Hendry, and Moran (2008) argue that the bias in inflation expectations can arise from small sample problems. In a recent study, using machine learning algorithms, Bianchi, Ludvigson, and Ma (2022) also document substantial bias in macroeconomic expectations of professional forecasters and reveal its cyclical properties. Relatedly, Azeredo da Silveira, Sung, and Woodford (2020) and Sung (2024) show that imprecise memory can lead to biased forecasts and have rich implications for sensitivity depending on forecasting horizons.[6]

The evidence of biased expectations extends beyond professional forecasters. Using the MSC, Bhandari, Borovička, and Ho (2023) document that households' inflation and unemployment rate forecasts feature pessimistic biases, which are counter-cyclical and co-move positively along the business cycle. Rozsypal and Schlafmann (2023) document a systematic bias in individual-level income expectations that varies with income levels.[7] They argue that an over-persistence bias rationalizes this evidence, which is broadly consistent with the approach proposed by Molavi (2023). Using survey evidence in UK, Michelacci and Paciello (2024) study how the pessimistic bias in inflation forecasts is related to households' wishes about inflation and nominal interest rates.

We contribute to this literature by presenting evidence on the joint behavior of bias and persistence of forecast errors across the income distribution, which cannot be easily rationalized by existing theories of expectation formation. Our theory provides a joint characterization of bias and sensitivity, directly addressing these empirical patterns.

Finally, our paper is related to the large literature on optimal policy under incomplete information (Adam, 2007; Lorenzoni, 2010; Paciello and Wiederholt, 2014; Amador and Weill, 2010; Angeletos and Lao, 2020; Angeletos and Sastry, 2021). This line of work has mainly focused on the responsiveness and cyclicality of the policy instrument when agents are subject to informational frictions, while the mistakes made by the private sector are mostly temporary. Relative to the existing litera-

[6] The notion of bias in Azeredo da Silveira, Sung, and Woodford (2020) and Sung (2024) is more related to the comparison between sensitivities under rational expectations and bounded rationality, while our notion of bias refers to the deviations of the unconditional mean of forecast errors from the rational benchmark.

[7] See Dominitz and Manski (1997), Dominitz (1998), Das and van Soest (1999), and Massenot and Pettinicchi (2019) for more related studies on the pessimistic bias of expectations on individual income.

ture, the additional bias and sensitivity in our environment introduce a trade-off between short-run responsiveness and long-run bias in policy design, and we illustrate the different welfare implications with alternative micro-foundations for the presence of biases.

2 An Illustrative Example

In this section, we discuss the interaction between ambiguity and imperfect information in a single-agent environment. We show how ambiguity aversion increases sensitivity to signals and leads to biased forecasts, relative to the Bayesian benchmark. We then extend these insights to a more general setting in the next section, where we introduce recurrent shocks and general equilibrium considerations.

2.1 A pure forecasting problem

Consider an inference problem about some exogenous economic fundamental. Assume the fundamental ξ is drawn from a Gaussian distribution with mean $\bar{\mu}$ and variance σ_ξ^2,

$$\xi \sim \mathcal{N}(\bar{\mu}, \sigma_\xi^2).$$

Agent i does not observe the fundamental perfectly, but receives a private noisy signal x_i about it:

$$x_i = \xi + \epsilon_i, \quad \text{with} \quad \epsilon_i \sim \mathcal{N}(0, \sigma_\epsilon^2).$$

In the absence of ambiguity, the distributions of underlying shocks are common knowledge. Agents perceive that there is a single model, or a single probability distribution, that describes the stochasticity of random variables. Uncertainty arises from the realization of shocks within a model, which is referred to as risk.

Bayesian expectations To begin, we revisit the standard Bayesian benchmark. In this case, agents simply want to minimize the mean-squared error (MSE) of their forecast. With Gaussian shocks, we can focus on linear strategies,

$$g(x_i) = sx_i + b,$$

where the s represents the sensitivity to signals and b is some constant. Given a particular strategy characterized by the pair (s, b), the MSE can be expressed as the sum of two terms

$$\mathbb{E}\left[(g(x_i) - \xi)^2\right] = \mathcal{R}(s) + (b - (1-s)\,\bar{\mu})^2, \quad \text{where} \quad \mathcal{R}(s) = s^2\sigma_\epsilon^2 + (1-s)^2\,\sigma_\xi^2. \tag{2.1}$$

The first term, $\mathcal{R}(s)$, represents the cost of within-model uncertainty (risk)—a weighted sum of the variance of the fundamental and of the noise. For any sensitivity s, the second term can always be set to zero by appropriately choosing the constant b. It follows that the optimal sensitivity, s^{RE}, is obtained by minimizing the cost of risk:

$$s^{\text{RE}} = \operatorname{argmin} \mathcal{R}(s) = \frac{\sigma_\xi^2}{\sigma_\xi^2 + \sigma_\epsilon^2}.$$

The optimal strategy is then given by

$$g(x_i) = s^{\text{RE}} x_i + (1 - s^{\text{RE}})\mu, \tag{2.2}$$

a simple weighted average between the prior mean and the signal, following Bayesian updating. The optimal sensitivity corresponds to the familiar signal-to-noise ratio.

2.2 Ambiguity and Ambiguity Aversion

Ambiguity When agents face ambiguity, they could perceive multiple plausible models that describe the economy, each corresponding to a different distribution of underlying shocks. This generalization accommodates the possibility that agents may have doubts about what is the right model of the economy. For instance, in the aftermath of the Pandemic, consumers may be uncertain about whether inflation will fluctuate around an average level of 2% or 4% in the upcoming years. Similarly, during the slow recovery from the Great recession, firms may wonder whether their sales growth will remain stagnant or rebound to its pre-recession level.

We restrict our attention to the case where agents face ambiguity about the prior mean of the aggregate fundamental ξ.[8] Objectively, ξ is distributed according to $\mathcal{N}(\bar{\mu}, \sigma_\xi^2)$. Subjectively, however, agents' priors do not necessarily coincide with the objective distribution. They believe there can be multiple possible prior means, which themselves follow a normal distribution centered around the objective mean:

$$\xi \sim \mathcal{N}(\mu, \sigma_\xi^2), \quad \text{where} \quad \mu \sim \mathcal{N}(\bar{\mu}, \sigma_\mu^2).$$

In this case, different models are indexed by different μ, while σ_μ^2 parameterizes the ex-ante uncertainty about the models. When $\sigma_\mu^2 = 0$, we return to the Bayesian benchmark. Without loss of generality, we assume that $\bar{\mu} = 0$. The ex-ante probability density function of the perceived

[8] We explore ambiguity about the variance of the fundamental in Appendix G.1. In this scenario, the equilibrium strategy exhibits higher sensitivity but zero bias. Additionally, in Appendix G.2, we investigate the possibility that agents perceive ambiguity about the variance of the noise. In this case, we find that the equilibrium strategy exhibits less sensitivity rather than more, but still exhibits zero bias.

distribution of μ satisfies

$$p(\mu) \propto \exp\left(-\frac{1}{2}\sigma_\mu^{-2}\mu^2\right).$$

Ambiguity Aversion We now specify the preference of agents towards ambiguity. Specifically, we are interested in the case where agents are ambiguity-averse, meaning they dislike ambiguity more than risk. To this end, given a strategy $g(x_i)$, similarly to Klibanoff, Marinacci, and Mukerji (2005), we adopt the following loss function $\mathcal{L}(g)$,

$$\mathcal{L}(g) = \phi^{-1}\left(\int_\mu \phi\Big(\mathbb{E}^\mu\left[(g(x_i)-\xi)^2 - \chi\xi\right]\Big)p(\mu)\mathrm{d}\mu\right). \tag{2.3}$$

The integral over μ reflects the fact that agents face additional uncertainty about the prior mean. For each μ, $\mathbb{E}^\mu[\cdot]$ denotes the mathematical expectations in an economy where the prior mean is given by μ. We assume that agents can commit to following the strategy $g(x_i)$, which is equivalent to adopting the smooth rule when updating preferences, ensuring dynamic consistency (Hanany and Klibanoff, 2009).[9]

There are two additional modifications relative to the previous forecasting problem. First, the transformation $\phi(\cdot)$ introduces an additional cost associated with ambiguity about μ. When $\phi(\cdot)$ is linear, the problem reduces to

$$\mathcal{L}(g) = \int_\mu \mathbb{E}^\mu\left[(g(x_i)-\xi)^2 - \chi\xi\right]p(\mu)\mathrm{d}\mu. \tag{2.4}$$

This corresponds to the *ambiguity neutral* case, where the uncertainty about μ is treated in the same way as the within-model uncertainty about ξ and ϵ_i. Namely, adding uncertainty about μ is equivalent to drawing a compound lottery.

The distinction between ambiguity and risk becomes meaningful when the transformation $\phi(\cdot)$ is convex. Then, agents incur additional losses when their perceived model is found to be incorrect, and agents are said to have ambiguity aversion. In what follows, we assume that $\phi(\cdot)$ takes a constant absolute ambiguity aversion (CAAA) form, that is,

$$\phi(x) = \frac{1}{\lambda}\exp(\lambda x), \tag{2.5}$$

where $\lambda \geq 0$ measures the degree of ambiguity aversion.

Second, the reduced-form quadratic loss function can be derived from a general decision making problem in which the optimal strategy is equivalent to forecasting the solution under complete (or

[9] More details are provided in the remark at the end of Section 3.1.

perfect) information—this is the approach we follow in the Section 3. Note that the payoff directly depends on the level of the fundamental, captured by the term $\chi\xi$. For example, firms benefit from higher TFP regardless of their production decisions, while consumers benefit from higher real income independent of their consumption-saving decisions. The direct dependence on the exogenous fundamental is common in economic problems but can usually be ignored as it is inconsequential when agents are ambiguity neutral or when there is no ambiguity. In contrast, this dependence leads to biased forecast when agents are ambiguity-averse.

Sensitivity and bias To see how ambiguity aversion modifies agents' strategies, it is useful to decompose the loss function into the costs due to risk and ambiguity. Given a linear strategy $g(x_i) = sx_i + b$,[10] this decomposition can be expressed as

$$\mathcal{L}(g) = \underbrace{\mathcal{R}(s)}_{\text{cost of risk}} + \underbrace{\frac{1}{\lambda}\log\int_{\mu}\exp\left(\lambda\left[\left(b-(1-s)\,\mu\right)^2-\chi\mu\right]\right)p(\mu)\mathrm{d}\mu}_{\text{cost of ambiguity}}. \tag{2.6}$$

First, ambiguity aversion leads to a higher sensitivity towards signals. Intuitively, a more diffused prior leads agents to rely more heavily on signals, and ambiguity aversion amplifies this effect. To see the forces more clearly, note that the cost of risk, $\mathcal{R}(s)$, remains the same as in equation (2.1). When there is no ambiguity, an agent can simply choose $s = s^{\mathrm{RE}}$ to minimize $\mathcal{R}(s)$. When uncertainty about the prior mean is present, a trade-off arises between the cost of risk and the cost of ambiguity. At one extreme, if an agent only wants to minimize the cost of ambiguity, they would set $s = 1$ to eliminate the impact of ambiguity on forecast errors. Striking a balance between the two types of cost implies an enhanced sensitivity towards signals.

Second, ambiguity aversion introduces a bias in agents' strategies and forecasts. In equation (2.6), the direct effect of the exogenous fundamental, $\chi\mu$, is not symmetric around zero. Consider the case where $\chi > 0$. The losses in a model indexed with a negative prior mean are higher than the gains in a model indexed with a positive prior mean. Similar to the idea of self-insurance, an agent finds it optimal to mitigate losses in "rainy days" by assigning more weight to models with a negative μ. This affects the choice of the constant b, leading to biased forecasts. Ex ante, this incentive makes it appear as if agents have a more pessimistic view of the world.

The adjustment of sensitivity and bias in an agent's strategy is driven by their payoffs, as they place more weight on models that generate higher expected costs due to ambiguity aversion. Simultaneously, the magnitude of these costs is determined by the chosen strategy. This interdependence leads to a fixed-point problem. According to the following proposition, the optimal strategy can be

[10] With ambiguity aversion but in the absence of strategic interactions, the unique optimal response is a linear function of signals. See Appendix D for more details.

understood as if agents faced no ambiguity but had a modified prior belief.

Proposition 2.1. *The optimal linear strategy is equivalent to the Bayesian expectation with a more diffused prior belief,* $\xi \sim \mathcal{N}(0, \tilde{\sigma}_\mu^2(s^*) + \sigma_\xi^2)$, *and a bias*

$$g(x_i) = s^* x_i + \mathcal{B},$$

where the sensitivity s^* *satisfies*

$$s^* = \frac{\tilde{\sigma}_\mu^2(s^*) + \sigma_\xi^2}{\tilde{\sigma}_\mu^2(s^*) + \sigma_\xi^2 + \sigma_\epsilon^2}, \qquad \textit{with} \qquad \tilde{\sigma}_\mu^2(s) \equiv \frac{\sigma_\mu^2}{1 - 2\lambda\sigma_\mu^2(1-s)^2}, \tag{2.7}$$

and the bias is given by

$$\mathcal{B} = -\chi\lambda\sigma_\mu^2(1 - s^*). \tag{2.8}$$

Let us unpack these expressions. First of all, given the additional variance, $\tilde{\sigma}_\mu^2(s^*)$, in the prior belief, the optimal sensitivity is identical to the one under Bayesian expectations (2.2). The determination of the optimal sensitivity, s^*, requires solving a fixed-point problem. In the presence of ambiguity aversion, agents increase their sensitivity to reduce the ambiguity cost. At the same time, however, the degree to which they want to penalize the ambiguity cost is endogenously determined by the employed sensitivity. The system (2.7) succinctly summarizes these forces.

Next, even though the ex-ante belief about the distribution of μ is centered around zero, agents behave as if the prior mean is biased, captured by $\mathcal{B}$. The "as-if" shift of the mean is increasing in how much agents directly care about the fundamental, the degree of ambiguity aversion, and the amount of ambiguity.

The presence of ambiguity aversion predicts a joint pattern of sensitivity and bias. In the sequel, we show that this prediction is robust to different information structures and preferences and is supported by survey evidence on expectations.

3 General Equivalence Result

In this section, we present our main theoretical results. We extend the intuition from the illustrating example to an environment with more general preferences that also accommodate strategic interactions between agents. In addition, we allow for flexible persistent information structures and learning dynamics. We show that the optimal action under ambiguity aversion is akin to a pure forecasting problem of the fundamental with a modified prior belief.

3.1 Environment

Objective environment The economy is populated by a continuum of agents indexed by i. Agents care about a common fundamental, ξ_t, which follows the stochastic process:

$$\xi_t = a(L)\eta_t, \quad \text{with} \quad \eta_t \sim \mathcal{N}(0, \sigma_\eta^2),$$

where $a(L)$ is a polynomial in the lag operator L, and η_t is the innovation to the aggregate fundamental.[11]

Agents receive dispersed information about the fundamental. The vector of signals observed by individual agent i every period is given by

$$x_{it} = m(L)\eta_t + n(L)\epsilon_{it}, \quad \text{with} \quad \epsilon_{it} \sim \mathcal{N}(0, \Sigma), \tag{3.1}$$

where $m(L)$ and $n(L)$ are polynomial matrices in L that determine the dynamics of the signal process, and ϵ_{it} is a vector of idiosyncratic noises that wash out in aggregate. Unlike in Section 2, the fundamental as well as the signals can be persistent, which implies that the entire history of signals will be relevant for inference. To ensure that the random variables are stationary and signals do not contain future information, we make the following standard assumption.

Assumption 1. *All elements of $a(L)$, $m(L)$, and $n(L)$ contain only L with non-negative powers and are square-summable.*

Ambiguity In the objective environment, η_t is normally distributed with mean zero. Subjectively, agents believe that η_t is drawn from a Gaussian distribution with the same volatility, σ_η^2, but there is uncertainty about its prior mean, denoted by μ_t. The ambiguity about ξ_t is captured by the perception that

$$\eta_t \sim \mathcal{N}(\mu_t, \sigma_\eta^2), \quad \text{and} \quad \mu_t \sim \mathcal{N}(0, \sigma_\mu^2). \tag{3.2}$$

As in Section 2, the value of σ_μ^2 determines the degree of ambiguity.

Preference towards risk We first specify preferences about within-model risk. As a baseline, we consider a class of economies with quadratic utility given by

$$u(k_{it}, K_t, \xi_t) = -\frac{1}{2}\left[(1-\alpha)\left(k_{it} - \xi_t\right)^2 + \alpha\left(k_{it} - K_t\right)^2\right] - \chi\xi_t - \frac{1}{2}\gamma\xi_t^2, \tag{3.3}$$

[11] So far, η_t is the only aggregate shock in the economy. We extend the analysis to allow for multiple aggregate shocks in Appendix B.3.

where k_{it} denotes agent i's action and K_t denotes the aggregate outcome of the economy,

$$K_t \equiv \int k_{it} \mathrm{d}i.$$

The first component of the utility function, $(1-\alpha)(k_{it}-\xi_t)^2 + \alpha(k_{it}-K_t)^2$, captures the payoff directly associated with the individual agent's action. Agents aim to align their actions with the exogenous fundamental and the aggregate outcome. The degree of strategic complementarity, controlled by the parameter α, influences the strength of general equilibrium considerations. The second component, $\chi\xi_t + \frac{1}{2}\gamma\xi_t^2$, captures the non-strategic impact of the fundamental on the agent's utility. This component only affects the agent's optimal strategy when there is ambiguity and agents are ambiguity-averse.

The utility specification in equation (3.3) can be viewed as a quadratic approximation of a generic utility function, $u(k_{it}, K_t, \xi_t)$, as specified in Angeletos and Pavan (2007).[12] This specification accommodates strategic interactions among agents and a flexible dependence on fundamentals, but excludes the dependence on $(K_t - \xi_t)^2$, K_t, and $K_t\xi_t$. In the language of Angeletos and Pavan (2007), specification (3.3) pertains to economies that are efficient under both complete and incomplete information. However, in general, the underlying economy may be inefficient, and in such cases, dependence on the aforementioned terms may arise. Nevertheless, our main observational equivalence result still applies in these cases, as discussed in Appendix B.2.

Preference towards ambiguity Agents are assumed to be ambiguity-averse, with preferences represented by the smooth model of ambiguity,

$$\phi^{-1}\left(\int_{\mu^t} \phi\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it}, K_t, \xi_t\right)\right]\right) p\left(\mu^t\right) \mathrm{d}\mu^t\right). \tag{3.4}$$

The expectation operator $\mathbb{E}^{\mu^t}\left[u\left(k_{it}, k_t, \xi_t\right)\right]$ denotes the ex-ante expected utility under the model indexed by $\mu^t \equiv \{\mu_t, \mu_{t-1}, \ldots\}$, and $p\left(\mu^t\right)$ denotes the prior belief about μ^t derived from (3.2). We continue to assume the functional form:

$$\phi(x) = -\frac{1}{\lambda}\exp(-\lambda x),$$

which permits the tractability of the inference problem.

Remark on ex-ante strategy Similar to the illustrative example from Section 2, each agent i chooses their preferred strategy, which is now a contingency plan denoted by $k_{it} = g(x_i^t)$, for every

[12] The approximation is centered around the non-stochastic steady state: either the deterministic or the ambiguous steady state depending on the model environment.

possible history of private signals, $x_i^t \equiv \{x_{it}, x_{it-1}, \ldots\}$. We assume that all agents can commit to following their strategies, determined ex ante, when taking actions ex post. It is equivalent to assuming that the conditional preferences of agents upon receiving a history of private signals, x_i^t, are dynamically consistent.[13] The smooth rule of updating proposed by Hanany and Klibanoff (2009) ensures dynamic consistency. Moreover, the allocation with commitment coincides with that of the ex-ante equilibrium defined in Hanany, Klibanoff, and Mukerji (2020), which is sequentially optimal when conditional preferences are updated using the smooth rule.[14]

Finally, we make the following assumption to ensure that the problem is well-defined.[15]

Assumption 2. $\gamma \geq 0$ *and* $\lambda\gamma\frac{\sigma_\mu^2}{\sigma_\eta^2}\mathbb{V}(\xi_t) < 1$.

3.2 Subjective Beliefs and Equilibrium

We begin by defining a Nash equilibrium in our environment.

Definition 3.1. *A Nash equilibrium is a strategy $g(x_i^t)$ such that $k_{it} = g(x_i^t)$ maximizes the objective (3.4), and the aggregate outcome is consistent with individual actions, $K_t = \int g(x_i^t)\mathrm{d}i$.*

We focus on linear strategies, with $g(x_i^t)$ being linear functions of the history of signals. The next proposition establishes that a linear Nash equilibrium always exists, so this focus should be seen as an equilibrium refinement rather than a restriction.[16]

Proposition 3.1. *A Nash equilibrium with linear strategies exists.*

Without ambiguity aversion, the problem reduces to a standard beauty contest, and the optimal strategy can be written as an average of the expected fundamental and the expected aggregate outcome, as in Morris and Shin (2002). With ambiguity aversion, a similar result holds, albeit the expectations need to be based on the endogenous subjective beliefs. Consider the first-order condition for maximizing (3.4) with respect to the individual action k_{it}:

$$\int_{\mu^t} \mathbb{E}^{\mu^t}\left[\frac{\partial u(k_{it}, K_t, \xi_t)}{\partial k_{it}} \mid x_i^t\right] \phi'\left(\mathbb{E}^{\mu^t}[u(k_{it}, K_t, \xi_t)]\right) p\left(\mu^t \mid x_i^t\right) \mathrm{d}\mu^t = 0. \tag{3.5}$$

[13] We restrict attention to the set of preferences that preserve closure, namely each member of that set remains in the set after updating. This property holds naturally for expected utility preferences with Bayesian updating.

[14] The conditional preference under the smooth rule is made explicit in equation (A.1) in Appendix A.

[15] The first restriction, $\gamma \geq 0$, ensures that in the non-stochastic steady state under complete information, the utility function is concave in the fundamental. The second assumption stipulates that the level of ambiguity or the degree of ambiguity aversion should not exceed a certain threshold. It ensures that the ex-ante objective (3.4) is finite for at least one strategy, so that the agent's choice set is non-empty (see the Proof of Lemma A.5 in Appendix A for a more detailed discussion).

[16] With strategic interactions, we cannot rule out the possibility that all agents collectively choose a non-linear strategy. That being said, as shown in Appendix D, when the coordination motive vanishes (as in Section 2), there exists a unique solution to the individual's forecasting problem which is linear in signals.

Notice that when evaluating the payoff implications of an action, relative to the Bayesian kernel $p(\mu^t|x_i^t)$, agents behave as if they were using an expectation kernel distorted by $\phi'(\mathbb{E}^{\mu^t}[u(k_{it}, K_t, \xi_t)])$, which we refer to as the agents' subjective beliefs. This distortion reflects the fact that, whenever a model generates lower ex-ante expected utility in equilibrium, agents would regard it as the more likely model in their posterior belief relative to the Bayesian posterior. Recall that we have already employed this line of argument when explaining the bias term in the illustrating example. In the special case in which agents are ambiguity neutral, $\phi'(\cdot) = 1$, and the subjective kernel coincides with the Bayesian one. The following proposition summarizes this discussion.

Proposition 3.2. *Taking the law of motion of K_t as given, agent i's best response satisfies*

$$k_{it} = (1-\alpha)\,\mathcal{F}_{it}[\xi_t] + \alpha\,\mathcal{F}_{it}[K_t], \tag{3.6}$$

where $\mathcal{F}_{it}[\cdot]$ denotes agent i's subjective expectation operator, that is

$$\mathcal{F}_{it}[\cdot] \equiv \int_{\mu^t} \mathbb{E}^{\mu^t}[\,\cdot \mid x_i^t]\,\hat{p}\left(\mu^t \mid x_i^t\right)\mathrm{d}\mu^t, \quad \textit{with} \quad \hat{p}\left(\mu^t \mid x_i^t\right) \propto \phi'\left(\mathbb{E}^{\mu^t}[u(k_{it}, K_t, \xi_t)]\right) p\left(\mu^t \mid x_i^t\right).$$

Importantly, since agents' payoffs depend on the aggregate outcome K_t, both the magnitude and dynamics of belief distortions hinge on the equilibrium coordination motive. Consequently, besides needing to anticipate the actions and beliefs of others, the way that agents form their beliefs is also affected by the aggregate outcome. The fact that the equilibrium outcome and the distorted subjective beliefs have to be jointly determined makes solving the equilibrium significantly more involved.

The equilibrium outcome can also be expressed as a weighted sum of higher-order subjective expectations.

Corollary 3.1. *In equilibrium, the aggregate outcome is a function of a weighted sum of infinite subjective higher-order expectations of ξ_t,*

$$K_t = (1-\alpha)\sum_{j=0}^{\infty} \alpha^j \overline{\mathcal{F}}_t^{j+1}[\xi_t], \tag{3.7}$$

where $\overline{\mathcal{F}}_t^1[\cdot] \equiv \int \mathcal{F}_{it}[\cdot]\mathrm{d}i$, and $\overline{\mathcal{F}}_t^{j+1}[\cdot] \equiv \int \mathcal{F}_{it}\left[\overline{\mathcal{F}}_t^j[\cdot]\right]\mathrm{d}i$.

This result indicates that when forming their own beliefs, agents must take into account the possible bias and altered sensitivity in others' forecasts, and this incentive is regulated by the degree of strategic complementarity. In Section 3.4, we show that this representation helps uncover the interaction between coordination and ambiguity aversion.

3.3 Equilibrium Characterization

In this subsection, we provide an equivalence result that circumvents the determination of higher-order subjective beliefs. We show that solving for the equilibrium strategy described above can be reduced to solving a single-agent Bayesian forecasting problem with a modified information structure. This forecasting problem can then be tackled using the Kalman filter, which facilitates the development of a convenient toolbox for solving models featuring ambiguity and persistent learning. The straightforward characterization it provides also allows us to derive valuable comparative statics results under general conditions.

Auxiliary forecasting problem First consider the following auxiliary inference problem about the fundamental with Bayesian forecasters, which we later link back to the economy with ambiguity.

Definition 3.2. *The (w, α)-modified signal process is given by*

$$\tilde{\xi}_t = a(L)\tilde{\eta}_t, \qquad \textit{with} \quad \tilde{\eta}_t \sim \mathcal{N}(0, (1+w)\sigma_\eta^2), \tag{3.8}$$

$$\tilde{x}_{it} = m(L)\tilde{\eta}_t + n(L)\tilde{\epsilon}_{it}, \qquad \textit{with} \quad \tilde{\epsilon}_{it} \sim \mathcal{N}(0, (1-\alpha)^{-1}\Sigma), \tag{3.9}$$

where w is a non-negative scalar and α is the degree of complementarity. Let the optimal Bayesian forecast be given by

$$\tilde{\mathbb{E}}_{it}[\tilde{\xi}_t] = p(L; w, \alpha)\, \tilde{x}_{it}. \tag{3.10}$$

This signal process imposes two modifications relative to the original process. The volatility of the innovation to the fundamental is amplified by a factor of $(1+w)$, and the covariance matrix of the idiosyncratic noise is amplified by $(1-\alpha)^{-1}$. Intuitively, ambiguity aversion leads to a more diffuse prior about η_t, while the coordination motive reduces agents' incentives to rely on private information. These two considerations from the original environment are captured by the modifications to the shock processes in this auxiliary forecasting problem.

Sensitivity and bias Next, we introduce a notion of aggregate sensitivity and bias in our multivariate setting, which helps characterize the equilibrium strategy.

Definition 3.3. *Define the aggregate sensitivity to signals as*

$$\mathcal{S} \equiv 1 - \frac{\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)}{\mathbb{V}\left(\xi_t\right)}, \tag{3.11}$$

and the bias as

$$\mathcal{B} \equiv \mathbb{E}[\xi_t] - \mathbb{E}[K_t]. \tag{3.12}$$

With multiple signals, the sensitivity to signals can no longer be measured by the loading on a particular signal. To better understand the definition in equation (3.11), notice that under perfect information, the aggregate outcome K_t simply mirrors the exogenous fundamental ξ_t, and $\mathcal{S} = 1$. When information is incomplete, however, agents receive noisy signals and their aggregate response is dampened, reducing $\mathcal{S}$. Moving to the definition of bias in equation (3.12), notice that without ambiguity, the unconditional mean of the aggregate outcome coincides with that of the fundamental, resulting in $\mathcal{B} = 0$, even if information is incomplete. Ambiguity aversion, however, can lead to a permanent bias in agents' actions.

These definitions for sensitivity and bias can also be viewed as the counterparts of the regression coefficients in the following equation:

$$\xi_t - K_t = \beta_0 + \beta_1 \xi_t + \text{residuals}, \tag{3.13}$$

where β_0 corresponds to the bias, $\mathcal{B}$, and β_1 corresponds to the reverse of the sensitivity, $1-\mathcal{S}$.[17] These two moments contain rich information about agents' subjective beliefs and about the expectation formation process. As in Section 2, the levels of bias and sensitivity are jointly determined in equilibrium.

Equilibrium strategy Despite the complex interactions between persistent information, coordination motives, and ambiguity aversion, the equilibrium strategy ultimately takes a relatively simple form. This form can be connected to the notion of aggregate sensitivity and bias. Let $\tau_\mu \equiv \sigma_\mu^2/\sigma_\eta^2$ be a normalized measure of the amount of ambiguity.

Proposition 3.3. *The linear strategy in equilibrium takes the following form*

$$g(x_i^t) = (1+r)p(L; w, \alpha)x_{it} + \mathcal{B}. \tag{3.14}$$

1. *The polynomial matrix $p(L; w, \alpha)$ is the Bayesian forecasting rule in (3.10) with the (w, α)-modified signal process and w satisfies*

$$w = \left[\frac{1}{\tau_\mu} - \lambda\left(1-\alpha\right)\left(\mathbb{V}\left(\xi_t - K_t\right) + \frac{\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)^2\left(1-\mathcal{S}\right)^2}{1-\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}\right)\right]^{-1} \geq \tau_\mu. \tag{3.15}$$

2. *The additional amplification, r, satisfies*

$$r = \gamma\frac{\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)}{1-\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}\frac{w}{1+w}\left(1-\mathcal{S}\right) \geq 0. \tag{3.16}$$

[17] When K_t stands for the consensus forecasts about ξ_t, regression (3.13) resembles the ones explored in Kohlhas and Walther (2021).

3. The level of bias, $\mathcal{B}$, satisfies

$$\mathcal{B} = \chi \frac{\lambda \tau_\mu \mathbb{V}(\xi_t)}{1 - \lambda \gamma \tau_\mu \mathbb{V}(\xi_t)} (1 - \mathcal{S}). \tag{3.17}$$

To understand this result, let us first ignore ambiguity ($\tau_\mu = 0$). In this case, the equilibrium strategy reduces to $p(L; 0, \alpha)$. This means agents' actions are equivalent to forming Bayesian expectations about ξ_t, with the volatility of idiosyncratic noise adjusted to accommodate coordination considerations. This equilibrium characterization bypasses the dependence on higher-order beliefs and includes the single-agent results from Huo and Pedroni (2020).

When ambiguity is present ($\tau_\mu > 0$), prior uncertainty looms larger. This effect is captured by the amplified variance of the fundamental, by a factor of $(1 + w)$, in equation (3.8). Following the new forecasting rule $p(L; w, \alpha)$, agents' reliance on their signals increases when making forecasts. This channel makes it appear as if agents are overconfident in their signals, as in Broer and Kohlhas (2022). Moreover, remember that the term $-\chi \xi_t - \frac{1}{2} \gamma \xi_t^2$ in the utility function captures the non-strategic impact of the fundamental on the agent's payoff. A positive γ introduces an additional reason for agents to react to signals: the dependence on ξ_t^2 intensifies the impact of extreme realizations of the prior mean μ_t on welfare. This pushes agents' actions further away from the Bayesian benchmark through its effect on r. This additional overreaction is similar to the diagnostic expectations explored in Bordalo, Gennaioli, Ma, and Shleifer (2020), but in our environment the magnitude of this overreaction is endogenously determined in equilibrium.[18]

Additionally, if $\chi \neq 0$, the equilibrium strategy exhibits a permanent bias. Keep in mind that agents' subjective beliefs, $\mathcal{F}_{it}[\cdot]$, assign more weight to models that generate lower ex-ante expected utility, leading to a pessimistic bias.[19] Condition (3.17) indicates that the magnitude of the bias $\mathcal{B}$ and the level of sensitivity $\mathcal{S}$ are endogenously connected and constrained by equilibrium conditions. It is worth noting that condition (3.17) does not imply a direct inverse relationship between sensitivity and bias, as both are ultimately functions of deep parameters in the model. For instance, in Section 2, we observe that when the degree of ambiguity aversion varies, both bias and sensitivity move in the same direction. The joint pattern of bias and sensitivity yields testable implications, which we explore in Section 4.

Computation Even without ambiguity, solving for the equilibrium presents a significant challenge due to the combination of persistent information and coordination motives. Agents' strategies are infinite-dimensional objects with the entire history of signals as state variables. Although a finite-

[18]It should be noted that the implied overreaction in our equilibrium is built on a different micro-foundation than those in Broer and Kohlhas (2022) and Bordalo, Gennaioli, Ma, and Shleifer (2020). These approaches are suitable for different applied settings and are not necessarily substitutes.

[19]Notice that the sign of the bias depends on the sign of χ.

state representation is possible when signals follow finite ARMA(p, q) processes (Woodford, 2003; Angeletos and La'O, 2010; Huo and Pedroni, 2020), it remains unclear whether these results could be extended to models with ambiguity aversion. The following corollary confirms they can:

Corollary 3.2. *When the fundamental and signals follow finite ARMA processes, the equilibrium strategy admits a finite-state representation.*

Proposition 3.3 shows that the finite-state representation of the equilibrium can be achieved with ambiguity-averse agents, persistent information, and general-equilibrium considerations. It also outlines a clear computation method:[20]

1. For a particular pair of (w, α), the Bayesian forecasting rule $p(L; w, \alpha)$ can be obtained using standard algorithms such as the Kalman filter.
2. The value of r can be determined using condition (3.16). Together with the equilibrium strategy (3.14), this leads to the outcome K_t and the sensitivity $\mathcal{S}$. At this stage, the bias can be ignored as it does not affect any of the terms in the formulas.
3. Condition (3.15) can then be used to iterate on the value of w until convergence.
4. Finally, the bias can be obtained from equation (3.17).

In summary, this complex, seemingly infinite-dimensional problem effectively reduces to a one-dimensional fixed-point problem about w. In Section 4, we leverage this result to characterize the dynamics of inflation forecasts under ambiguity, which boils down to solving a single cubic equation.

3.4 Role of General Equilibrium Considerations

How do GE considerations interact with ambiguity aversion? To answer this question, it is useful to revisit the higher-order expectation representation in equation (3.7). A change in α not only affects the responses of expectations at each order, but also shifts the relative weight assigned to each expectation. However, these intensive and extensive margins do not necessarily move in the same direction in shaping equilibrium outcomes.

To illustrate the intuition with, we start with a simple example with static information, and then extend the findings to general information processes. Suppose that the utility function is given by

$$u\left(k_i, K, \xi\right) = -\frac{1}{2}\left[\left(1-\alpha\right)\left(k_i-\xi\right)^2 + \alpha\left(k_i-K\right)^2\right] - \chi\xi,$$

[20] We could also accommodate the case where the signals contain endogenous aggregate variables. Since each agent behaves in a competitive way, they still treat the information as exogenous even if signals are endogenous. Therefore, adding endogenous signals amount to adding another layer of fixed point problem on top of our equivalence result.

and that agents perceive ambiguity about the fundamental according to

$$\xi \sim \mathcal{N}\left(\mu, \sigma_\xi^2\right), \quad \text{and} \quad \mu \sim \mathcal{N}\left(0, \sigma_\mu^2\right),$$

where μ is objectively zero. Agents also observe a private signal $x_i \sim \mathcal{N}\left(\xi, \sigma_\epsilon^2\right)$. In this simple setup, we can further evaluate the effects of GE considerations on sensitivity and bias, by examining the properties of subjective higher-order expectations.

Proposition 3.4. *In the example economy, subjective higher-order expectations obey the following structure*

1. *The m-th order subjective expectation defined in condition (3.7) is given by*

$$\overline{\mathcal{F}}^m[\xi] = \kappa_m \xi + \beta_m, \quad \textit{with} \quad \kappa_m = \left(\frac{(1+w)\sigma_\xi^2}{(1+w)\sigma_\xi^2 + \sigma_\epsilon^2}\right)^m, \quad \textit{and} \quad \beta_m = \beta_{m-1} + (\kappa_m - \kappa_{m-1})\lambda\chi\sigma_\mu^2;$$

2. *The endogenous multiplier w is not monotonic in α;*
3. *The aggregate outcome is given by*

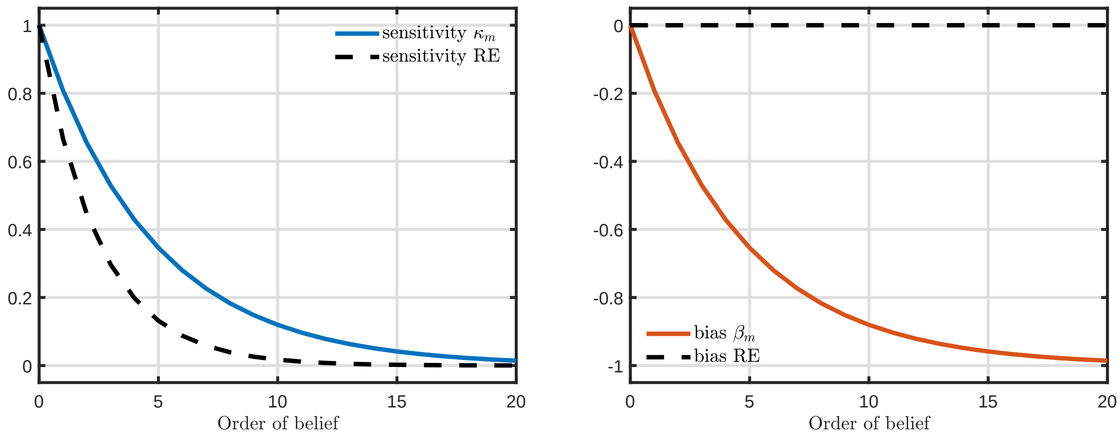
$$K = \mathcal{S}\,\xi + \mathcal{B} = (1-\alpha)\sum_{m=0}^{\infty}\alpha^m \kappa_m\,\xi + (1-\alpha)\sum_{m=0}^{\infty}\alpha^m \beta_m.$$

Part 1 implies that for a fixed α, as the order increases, the sensitivity of the subjective higher-order expectations, κ_m, decreases with the order. This is displayed in the left panel of Figure 3.1 and resembles the rational expectations result without ambiguity (Morris and Shin, 2002). Interestingly, the right panel of Figure 3.1 shows that the bias, β_m, is increasing in the order of expectation. This is a result of the accumulation of pessimism in ascending the hierarchy of beliefs—a footprint of ambiguity aversion. When forecasting the beliefs of others, agents internalize the bias of others in addition to their own.

Part 2 suggests that there are competing forces shaping the effect of a change in α on w. Ceteris paribus, a higher α shifts agents' attention from forecasting the fundamental to forecasting others' actions. As a result, private information about the fundamental becomes less relevant, and agents reduce how much they respond to signals. As can be seen in condition (3.15), the endogenous amplification force captured by w vanishes when α approaches 1.

However, this is not the only force at play. Condition (3.15) also reveals that a reduction in the sensitivity $\mathcal{S}$ could contribute to a higher w. This is due to the fact that, when agents reduce the responsiveness to signals, they also increase their reliance on their ambiguous prior, and this is costly

FIGURE 3.1: Subjective Higher-Order Beliefs



(A) Sensitivity of HOB (B) Bias of HOB

Note: This figure reports sensitivity (Panel A) and bias (Panel B) associated with the entire belief hierarchy as a function of the order of beliefs.

since agents are ambiguity-averse. Endogenously, this puts upwards pressure on w to undo this effect. Overall, these two forces leave the comparative statics of w with respect to α ambiguous. As shown in Figure 3.2, w exhibits a humped shape with respect to the strength of GE considerations.

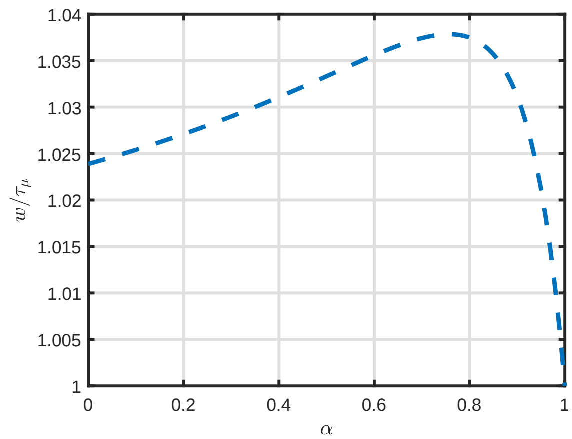
This nonmonotonicity is inherited by both κ_m and β_m. Together with Part 1 of Proposition 3.4, this implies that sensitivities and biases at every order mirror the pattern of w and are not monotonic with respect to the GE consideration α.

Part 3 states that the observed sensitivity and bias of the aggregate outcome is a weighted average of κ_m and β_m. Mechanically, as α increases, a larger weight is assigned to higher-order expectations relative to first-order expectations.

Ultimately, when α changes, the reweighting channel dominates the ambiguous effects on w, leading to overall lower sensitivity and increased bias. These observations hold beyond this simple example. Taking advantage of Proposition 3.3, we obtain the following result for a general information structure.

Proposition 3.5. *When $\gamma = 0$, the sensitivity $\mathcal{S}$ is decreasing in α, and the magnitude of bias $|\mathcal{B}|$ is increasing in α.*

Figure 3.2: Comparative Statics of w over α.



3.5 Extensions

The results discussed so far depend on assumptions about the payoff function and the information structure. However, the observational equivalence result and the insights regarding bias and sensitivity are more general.

Multiple actions In the baseline specification, each agent chooses only one action, while firms and households often need to simultaneously consider multiple choices. For example, a manager must jointly plan hiring and advertisement expenditures; a consumer must decide on the allocation of time for work-related activities, educational activities, household work, and so on. We now extend our baseline specification to accommodate this possibility.

We maintain the specification for the shocks and ambiguity but modify the preference towards risk as follows

$$u(k_{it}, K_t, \xi_t) = \frac{1}{2}\left(k_{it} - \kappa\xi_t\right)' \Psi_k \left(k_{it} - \kappa\xi_t\right) + \frac{1}{2}\left(k_{it} - K_t\right)' \Psi_K \left(k_{it} - K_t\right) + \chi\xi_t - \frac{1}{2}\gamma\xi_t^2, \qquad (3.18)$$

which is a generalization of the univariate case (3.3) to allow for multiple actions.[21] Here, the individual actions, k_{it}, and the aggregate actions, K_t, are both $J \times 1$ random vectors. The constant $J \times J$ matrices Ψ_k and Ψ_K contain information on how individual and aggregate actions affect welfare in the steady state. Finally, κ is a $J \times 1$ vector of constants that determines how agents would respond to the fundamental in a full information economy.

Unlike the baseline environment, an individual agent's best response leads to a multivariate beauty

[21]This specification is derived from a quadratic approximation of generic preferences, which is efficient under incomplete information. See Appendix B.1 for details.

contest game.

Lemma 3.1. *Denote subjective expectations by $\mathcal{F}_{it}[\cdot]$ as in Proposition 3.2. The best response satisfies*

$$k_{it} = (I - \Theta)\, \mathcal{F}_{it}\left[\kappa\xi_t\right] + \Theta \mathcal{F}_{it}\left[K_t\right], \quad \textit{with} \quad \Theta \equiv (\Psi_k + \Psi_K)^{-1}\Psi_K.$$

In addition, denote the eigenvalue decomposition of Θ as

$$\Theta = Q^{-1} diag(\alpha_1, \ldots, \alpha_J) Q.$$

The best response system can be transformed into

$$Qk_{it} = (I - diag(\alpha_1, \ldots, \alpha_J))\, \mathcal{F}_{it}\left[Q\kappa\xi_t\right] + diag(\alpha_1, \ldots, \alpha_J)\mathcal{F}_{it}\left[QK_t\right].$$

With multiple actions, the interdependence between actions makes strategic complementarities more involved, summarized now by the matrix Θ. When information is perfect and there is no ambiguity about the fundamental, $k_{it} = K_t = \kappa\xi_t$, and the matrix Θ is irrelevant. When information is incomplete, Θ starts to play a role.

The second part of Lemma 3.1 shows that the multivariate system can be transformed into a seemingly decoupled system: the vector of transformed actions, Qk_{it}, is orthogonal in the sense that the corresponding complementarities are now represented by a diagonal matrix. This is indeed the case when the expectation operator is independent of agents' actions. However, when agents are ambiguity-averse, subjective beliefs depend on expected utility. As a result, the actions remain interconnected through the endogeneity of the beliefs.

To proceed, we define bias and sensitivity for the multiple action case, now given by $J \times 1$ vectors,

$$\mathcal{B} \equiv \mathbb{E}[\kappa\xi_t] - \mathbb{E}[K_t], \quad \text{and} \quad \mathcal{S} \equiv \mathbf{1} - \frac{\mathbb{COV}\left(\kappa\xi_t - K_t, \xi_t\right)}{\mathbb{V}\left(\xi_t\right)}.$$

We also introduce the $J \times J$ matrix W that generalizes the scalar w from condition (3.15),

$$W \equiv \left(\tau_\mu^{-1}\mathrm{I} + \lambda Q\left(\mathbb{V}\left(\kappa\xi_t - K_t\right) + \frac{\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)^2}{1 - \lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}\left(\mathbf{1} - \mathcal{S}\right)\left(\mathbf{1} - \mathcal{S}\right)'\right)\Psi_k Q^{-1}\right)^{-1}.$$

The following proposition circumvents the aforementioned difficulties and establishes an equivalence result that resembles the univariate one.

Proposition 3.6. *The linear strategy in equilibrium takes the following form*[22]

$$k_{it} = g(x_i^t) = Q^{-1}P^{-1}\begin{bmatrix} \left(\sum_{j=1}^{J} P_{1j}\left(1+\frac{(1-\alpha_j)}{(1-\alpha_1)}r_j\right)e_j'Q\kappa\right)p(L;w_1,\alpha_1) \\ \vdots \\ \left(\sum_{j=1}^{J} P_{Jj}\left(1+\frac{(1-\alpha_j)}{(1-\alpha_J)}r_j\right)e_j'Q\kappa\right)p(L;w_J,\alpha_J) \end{bmatrix} x_{it} + \mathcal{B}.$$

1. *The polynomial matrix $p(L;w_j,\alpha_j)$ is the Bayesian forecasting rule in (3.10). The matrix P and the scalars w_j derive from the following eigenvalue decomposition*

$$(\mathrm{I} - diag(\alpha_1,\ldots,\alpha_J))\left(\tau_\mu^{-1}\mathrm{I} + W\right) = P^{-1}diag(\omega_1,\ldots,\omega_J)P, \quad and \quad w_j \equiv \frac{\tau_\mu\omega_j}{1-\alpha_j} - 1.$$

2. *The additional amplification, r_j, satisfies*

$$r_j = \gamma\frac{\lambda\tau_\mu\mathbb{V}(\xi_t)}{1-\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}\frac{e_j'WQ(\mathbf{1}-\mathcal{S})}{e_j'Q\kappa_1(1+w_j)}.$$

3. *The bias vector, $\mathcal{B}$, satisfies*

$$\mathcal{B} = \chi\frac{\lambda\tau_\mu\mathbb{V}(\xi_t)}{1-\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}(\mathbf{1}-\mathcal{S}).$$

The equilibrium strategy is effectively a linear combination of J different Bayesian forecasts of the fundamental. In these modified forecasts, the idiosyncratic noises are adjusted according to the eigenvalues of the matrix of complementarities, Θ, which is exogenous. On the other hand, the shocks to fundamentals are adjusted according to the eigenvalue of an endogenous matrix involving W. The matrix P provides the required rotation, on top of the matrix Q, so that the vector of actions can be analyzed in an orthogonal way, even when ambiguity aversion is present. Notably, the biases for each action share a common component but also depend on their specific correlation with the fundamental, captured by the vector κ. This allows the bias to vary in direction for different actions.

Other extensions In Appendix B.2, we allow the economy to be inefficient under both complete and incomplete information. Essentially, our equivalence result applies to general quadratic payoff structures. Our optimal-policy application in Section 5 utilizes this set of results.

In another extension, Appendix B.3 shows how to generalize these insights to settings with multiple aggregate shocks, including common noises. This extension can be used to study the interaction between ambiguity and the effects of non-fundamental driven fluctuations, for example.

[22]The vector e_j denotes the j-th column of the $J \times J$ identity matrix.

Finally, Appendix F explores the effects of ambiguity on the value of increasing signal precision. The answer to this question is particularly relevant for models of rational inattention, where signal precision is chosen endogenously, balancing the cost and benefit of acquiring information. We show that the value of higher precision in signals is increasing in the amount of ambiguity.

4 Application: Inflation Forecasts

Under full information rational expectations (FIRE), optimal forecasts are unbiased and forecast errors are serially uncorrelated. In contrast, using survey data on expectations, the literature has demonstrated that the average forecasts of agents are biased and that forecast errors are serially correlated.[23] In this section, we document additional survey evidence on the joint behavior of bias and persistence across the income distribution, which cannot be easily rationalized by existing theories of expectation formation. We then show that these facts emerge naturally from a micro-founded model with ambiguity-averse consumers.

4.1 Data and Facts

The Michigan Survey of Consumers (MSC) collects data on household inflation expectations asking what is their "price expectations for the next 12 months." It also provides information about household income, which allows us to allocate surveyed households in each quarter into groups based on their income percentiles. For each income group g, the average inflation forecast error in quarter t is calculated as the average inflation forecast error across every household $i \in \mathbb{I}_g$:

$$\overline{\mathrm{FE}}_{g,t} \equiv \int_{\mathbb{I}_g} \left(\pi_{t,f} - \mathcal{F}_{i,g,t}[\pi_t]\right) \mathrm{d}i,$$

where we continue to use $\mathcal{F}_{i,g,t}$ to denote households' subjective expectations. The bias and persistence of forecast errors for each income group are given by their across-time average and autocorrelation:

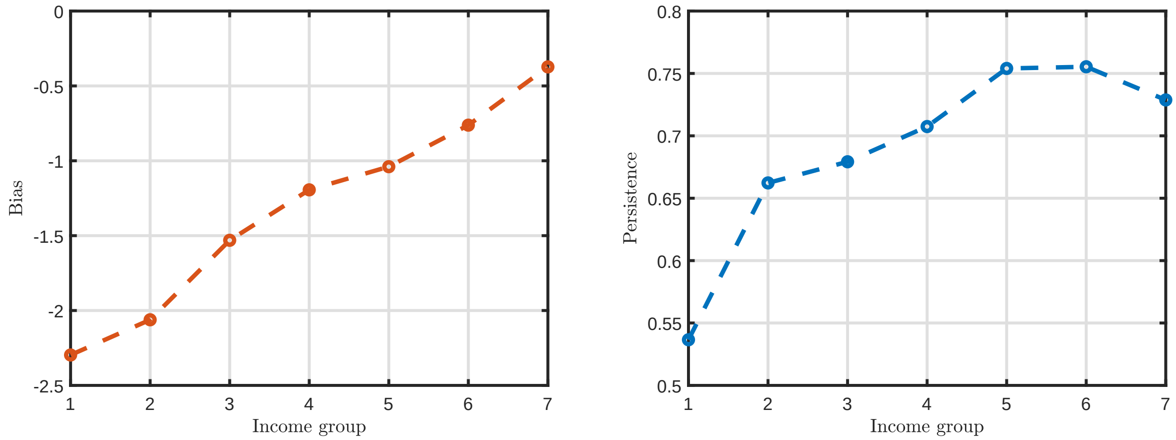
$$\mathrm{Bias}_g \equiv \frac{1}{T}\sum_{t=1}^{T} \overline{\mathrm{FE}}_{g,t}, \qquad \text{and} \qquad \mathrm{Persistence}_g \equiv \mathrm{Corr}\left(\overline{\mathrm{FE}}_{g,t}, \overline{\mathrm{FE}}_{g,t-1}\right). \tag{4.1}$$

Figure 4.1 presents the bias (panel A) and persistence (panel B) of households' one-year-ahead inflation forecasts. In line with the existing literature, households' inflation forecasts are biased upwards leading to negative average forecast errors. What is interesting is the joint behavior of bias and persistence in the cross-section of household incomes. As households move up the income ladder, the bias in their inflation expectations decreases, while the persistence of their inflation forecast errors

[23] See, for example, Farmer, Nakamura, and Steinsson (2023) for evidence of bias and autocorrelated forecast errors in the survey of professional forecasters; Kohlhas and Robertson (2024) for evidence of bias in professional forecasters' forecasts, and Bhandari, Borovička, and Ho (2023) for evidence of bias in household forecasts.

increases.

FIGURE 4.1: Bias and Persistence of Forecast Error in the Survey Data



(A) Bias in forecast

(B) Persistence of forecast error

Note: This figure reports bias (Panel A) and persistence (Panel B) of households' inflation forecasts in the cross-section of the income distribution. Bias and persistence of each income percentile are calculated by the mean and serial correlation of forecast errors of households' inflation expectations for the next 12 months. Data are obtained from the Michigan Survey of Consumers between 1987:I and 2020:IV.

In Appendix H.1, we further document several additional results: (1) In the Survey of Consumer Expectations (SCE), conducted by the Federal Reserve of New York, the joint pattern of bias and persistence of forecast errors is similar to the one observed in the MSC, as shown in Table H.1. (2) At the individual level, after controlling for other observed characteristics such as age and resident state, the magnitude of the bias continues to decrease with income, as shown in Table H.2.

The documented patterns of bias and persistence of forecast errors present challenges to the assumption of rational expectations. Within the rational expectations paradigm, whether under full or noisy information, forecast errors should be zero on average. Noisy rational expectations can generate persistent forecast errors, but matching the increasing pattern of persistence across household income would require having richer households being less informed about the economy. A notable exception is Farmer, Nakamura, and Steinsson (2023). They demonstrate that slow learning over the long-run trend with a unit root can create bias and persistence in forecast errors. However, the question of why bias and persistence move in opposite directions as household income changes remains unaddressed.

In what follows, we set up a two-period consumption model with sticky nominal income and ambiguity about the inflation process. Applying the results developed in Section 3, we show that the inflation forecasts that arise from this model are consistent with the type of cross-sectional distribution of bias and persistence documented above.

4.2 Model

Household problem There is a finite number of household groups indexed by $g \in 1, \ldots, N$, each differing in their nominal income level, denoted by Y_g. Within each income group, there is a continuum of households indexed by i. We consider a simple, stylized consumption-saving problem where households plan their consumption only for periods t and $t+1$. The utility function of the household is given by

$$U\left(C_{i,g,t}, C_{i,g,t+1}\right) = \frac{C_{i,g,t}^{1-\nu} - 1}{1-\nu} + \beta \frac{C_{i,g,t+1}^{1-\nu} - 1}{1-\nu},$$

where ν controls the degree of risk aversion of households.

Nominal income is rigid between t and $t+1$. A household in group g receives nominal income $P_t Y_g/2$ in each period. Therefore, their budget constraint is given by

$$P_t C_{i,g,t} + P_{t+1} C_{i,g,t+1} = P_t Y_g.$$

Let $\pi_{t+1} \equiv (P_{t+1} - P_t)/P_t$ denote the inflation rate. The budget constraint for household i can then be rewritten as

$$C_{i,g,t} + \left(1 + \pi_{t+1}\right) C_{i,g,t+1} = Y_g.$$

That is, a higher inflation rate makes consumption tomorrow more expensive relative to consumption today, adversely affecting households due to the nominal rigidity in their income.[24] In this problem, households only face uncertainty about the inflation rate. Once the belief about future inflation is determined, the optimal consumption plan follows. The following lemma directs us to concentrate on inflation expectations.

Lemma 4.1. *Around the zero-inflation steady state:*

1. *The optimal consumption change, $c_{i,g,t}$, is proportional to the household's subjective expectation about inflation*

$$c_{i,g,t} = \frac{\beta^{1/\nu}}{1+\beta^{1/\nu}} \mathcal{F}_{i,g,t}\left[\pi_{t+1}\right]; \tag{4.2}$$

[24] The assumption of complete nominal income rigidity is made only for simplicity. What is important is that, relative to goods prices, nominal income changes at a slower rate.

2. *The quadratic approximation of the utility function $U(C_{i,g,t}, C_{i,g,t+1})$ is given by*

$$U \approx Q(\mathcal{F}_{i,g,t}[\pi_{t+1}], \pi_{t+1}) = const - \delta_g (\mathcal{F}_{i,g,t}[\pi_{t+1}] - \pi_{t+1})^2 - \chi_g \pi_{t+1} - \gamma_g \pi_{t+1}^2, \quad (4.3)$$

where δ_g, χ_g, and γ_g are positive, when $\nu > 1$, and satisfy

$$\delta_g, \chi_g, \gamma_g \propto Y_g^{1-\nu}.$$

The first part of Lemma 4.1 states that there is a one-to-one mapping between the optimal choice of consumption and the household's subjective expectation about the inflation rate. A higher inflation expectation implies that future goods become more expensive relative to current goods, and households have an incentive to increase their current consumption.

Condition (4.2) allows us to express the utility in terms of expected inflation and actual inflation, since future consumption depends on the realized inflation. This quadratic approximation of the utility function is nested within the general specification (3.3) in Section 3.1. The utility function (4.3) reveals two important properties that are crucial for matching the data later on: (1) ceteris paribus, higher inflation lowers household welfare since both χ_g and γ_g are positive; (2) a higher level of income reduces households' exposure to variations in inflation since $(\delta_g, \chi_g, \gamma_g)$ are decreasing in Y_g.

Subjective expectations The inflation rate, π_t, follows an exogenous AR(1) process,

$$\pi_t = \rho \pi_{t-1} + \eta_t, \quad \text{with} \quad \eta_t \sim \mathcal{N}(0, \sigma_\eta^2),$$

and at each period t, household i receives a noisy private signal,

$$x_{i,g,t} = \pi_t + \epsilon_{i,g,t}, \quad \text{with} \quad \epsilon_{i,g,t} \sim \mathcal{N}(0, \sigma_\epsilon^2).$$

The information set of household i at time t is $\mathcal{I}_{i,g,t} = \{x_{i,g,t}, x_{i,g,t-1}, \ldots\}$. So far, this specification is similar to the structure used in the literature that studies survey evidence on inflation forecasts (Coibion and Gorodnichenko, 2015; Bordalo, Gennaioli, Ma, and Shleifer, 2020). It is also worth noting that the received information is independent of the household's income level.

We depart from rational expectations by allowing agents to perceive ambiguity about the mean of the innovation to inflation,

$$\eta_t \sim \mathcal{N}(\mu_t, \sigma_\eta^2), \quad \text{and} \quad \mu_t \sim \mathcal{N}(0, \sigma_\mu^2),$$

where, again, σ_μ^2 corresponds to the amount of ambiguity. Let $h(x_{i,g}^t)$ denote the households' strategy in forming expectations, i.e., $\mathcal{F}_{i,g,t}[\pi_{t+1}] = h(x_{i,g}^t)$. The mapping from subjective inflation expectations to optimal consumption in period t allows us to transform the original optimal consumption problem into an optimal forecasting problem that can be embedded into our general theoretical framework. Following the specification from Section 3, households maximize the following objective function:

$$\phi^{-1}\left(\int_{\mu^t}\phi\left(\mathbb{E}^{\mu^t}_{i,g,t}\left[Q\left(h\left(x^t\right),\pi_{t+1}\right)\right]\right)p\left(\mu^t\right)\mathrm{d}\mu^t\right). \tag{4.4}$$

By establishing the equivalence between the optimal forecasting problem (4.4) and the original optimal consumption problem, we are implicitly assuming that households respond to the survey questions about inflation by reporting beliefs that are consistent with their consumption decisions. This assumption is common in studies that connect survey data on expectations to structural models with ambiguity-averse agents, such as Bhandari, Borovička, and Ho (2023) and Pei (2023).

4.3 Inflation Forecasts under Ambiguity

Bayesian benchmark We begin by muting ambiguity ($\sigma^2_\mu = 0$) and considering the predictions of this model under standard Bayesian expectations, characterized in the following proposition.

Proposition 4.1. *The Bayesian forecast is a weighted sum of the prior and the new signal,*

$$\mathbb{E}_{i,g,t}[\pi_{t+1}] = \omega\,\mathbb{E}_{i,g,t-1}[\pi_t] + (\rho-\omega)x_{i,g,t}, \tag{4.5}$$

and the average forecast error satisfies

$$\pi_{t+1} - \overline{\mathbb{E}}_{g,t}[\pi_{t+1}] = \frac{1}{1-\omega L}\eta_{t+1},$$

where the persistence ω is given by

$$\omega = \frac{1}{2}\left(\rho + \frac{\sigma^2_\epsilon + \sigma^2_\eta}{\rho\sigma^2_\epsilon} - \sqrt{\left(\rho + \frac{\sigma^2_\epsilon + \sigma^2_\eta}{\rho\sigma^2_\epsilon}\right)^2 - 4}\right).$$

Without ambiguity, households underreact to their signals due solely to the noise in their observations. As a result, the aggregate forecast errors are persistent over time, as discussed in Coibion and Gorodnichenko (2015). However, the persistence ω does not correlate with household income levels. Moreover, the unconditional mean of household forecasts coincides with that of actual inflation, leaving no room for permanent bias. These properties are inconsistent with the empirical facts documented in Section 4.1.

Ambiguity-averse households When households are ambiguity-averse, their income level matters for their subjective beliefs. The following characterization directly applies the results developed in Section 3.

Proposition 4.2. *With ambiguity aversion, the individual subjective forecast satisfies*

$$\mathcal{F}_{i,g,t}[\pi_{t+1}] = \vartheta_g \mathcal{F}_{i,g,t-1}[\pi_t] + (1+r_g)(\rho - \vartheta_g)x_{i,g,t} + (1-\vartheta_g)\,\mathcal{B}_g, \tag{4.6}$$

and the average forecast error obeys

$$\pi_{t+1} - \overline{\mathcal{F}}_{g,t}[\pi_{t+1}] = \frac{1+r_g}{1-\vartheta_g L}\eta_{t+1} - \frac{r_g}{1-\rho L}\eta_{t+1} - \mathcal{B}_g,$$

where the persistence ϑ_g is given by

$$\vartheta_g = \frac{1}{2}\left(\rho + \frac{(1+w_g)\sigma_\eta^2 + \sigma_\epsilon^2}{\rho\sigma_\epsilon^2} - \sqrt{\left(\rho + \frac{(1+w_g)\sigma_\eta^2 + \sigma_\epsilon^2}{\rho\sigma_\epsilon^2}\right)^2 - 4}\right) < \omega, \tag{4.7}$$

with $r_g > 0$, $w_g > 0$, and $\mathcal{B}_g > 0$.

Condition (4.6) shows that the subjective inflation expectation follows a law of motion similar to the Bayesian one but with several important modifications. First, households consistently overestimate the inflation rate by an amount determined by the bias term $\mathcal{B}_g$.

Second, the persistence of forecast errors is smaller than in the Bayesian case, $\vartheta_g < \omega$, for all g. Households react to their signals as if they are overconfident à la Broer and Kohlhas (2022), as the perceived signal-to-noise ratio $(1+w_g)\sigma_\eta^2/\sigma_\epsilon^2$ is larger. Thus, households rely less on their prior means, which reduces the persistence of their forecasts.

Third, households exhibit an additional overreaction to their current signals relative to a Bayesian rule, captured by the term $1+r_g$. The forecasting rule (4.6) shares similar properties to the one under diagnostic expectations (Bordalo, Gennaioli, Ma, and Shleifer, 2020) in which forecasters overweight representative states. This additional response to the signal distinguishes it from a Bayesian rule.

In the setup with ambiguity, the magnitudes of overreaction and bias endogenously depend on how inflation enters households' payoff functions. As emphasized in Lemma 4.1, the parameters governing a household's exposure to inflation decrease with their income level. This reduced exposure is isomorphic to a lower degree of ambiguity aversion. As a result, higher-income households are effectively less concerned about ambiguity, and their forecasting strategy is more aligned with the Bayesian benchmark. It follows that, qualitatively, richer households are less biased and their forecast errors are more persistent, which is consistent with the patterns observed in the data.

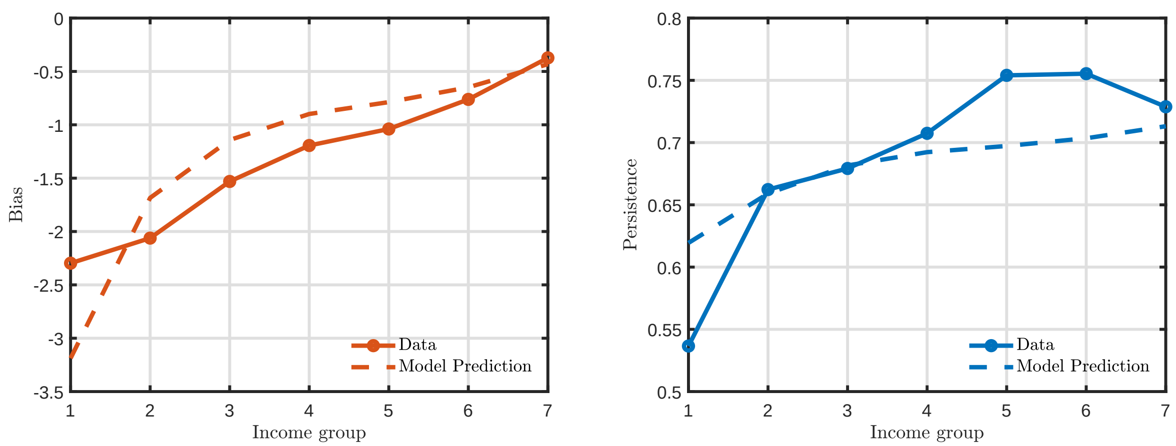
4.4 Results

To bring the model to the data, we set $\rho = 0.88$ and $\sigma_\eta = 0.72$ to fit the actual inflation process, and fix the household's discount factor to $\beta = 0.99$. We normalize average income to 1, and set Y_g to match the share of income of each group in the MSC. The remaining parameters are the standard deviation of private information, σ_ϵ, the amount of ambiguity, σ_μ, the degree of ambiguity aversion, λ, and the degree of relative risk aversion, ν. We calibrate these parameters to match the persistence and bias of inflation forecast errors displayed in Figure 4.1. Operationally, we minimize

$$\sum_g \left(\text{Bias}_g^{\text{data}} - \text{Bias}_g^{\text{model}}\right)^2 + \left(\text{Persistence}_g^{\text{data}} - \text{Persistence}_g^{\text{model}}\right)^2 .$$

The bias and persistence in the data are computed from the MSC using equation (4.1), and their model counterparts are the theoretical moments derived from Proposition 4.2.

Figure 4.2: Goodness of Fit.



(A) Bias in forecast (B) Persistence of forecast error

Table 4.1 presents the calibrated parameters. The model fit requires both signal noise and perceived ambiguity. Figure 4.2 displays the goodness of fit of our calibrated model. Given its rather parsimonious structure, our model captures the cross-sectional patterns of bias and persistence of inflation forecast errors reasonably well: richer households tend to exhibit less bias in their inflation forecasts but more persistent forecast errors.

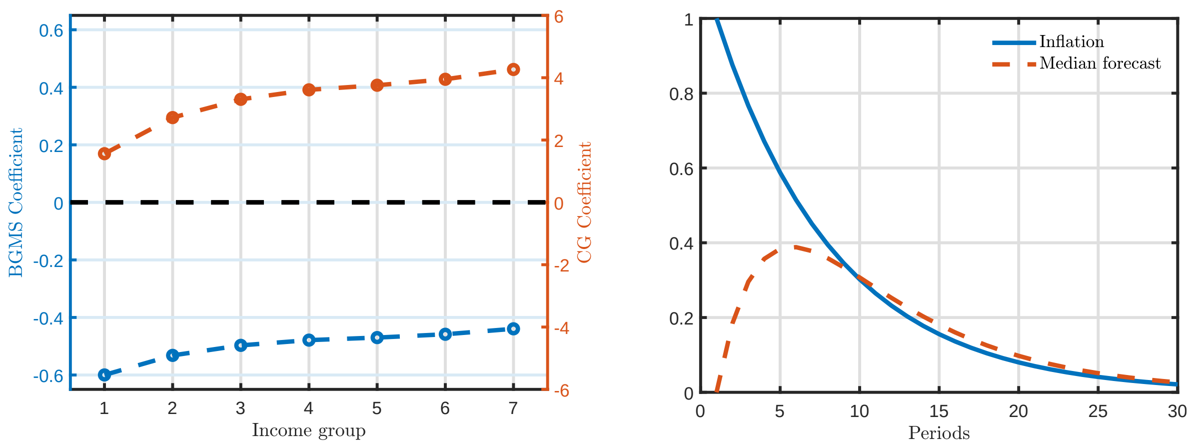
It is challenging to capture these cross-sectional patterns in the Bayesian-expectation model without ambiguity aversion, where bias is zero and persistence does not vary with income. If households perceive ambiguity ($\sigma_\mu > 0$) but their preference is ambiguity-neutral ($\lambda = 0$), the persistence of

Table 4.1: Calibrate Parameters

Param.	Value	Related to
σ_ϵ	4.11	std of noise in private signals
σ_μ	0.82	amount of ambiguity
λ	0.23	degree of ambiguity aversion
ν	1.46	risk aversion

forecast errors would increase with income, but bias would still not be present.

Figure 4.3: Conditional and Unconditional Moments of Subjective Beliefs



(A) CG and BGMS coefficients (B) IRF of median forecast

Note: Panel A reports the theoretical CG and BGMS regression coefficients for each income group. Panel B reports the theoretical impulse response of the the consensus inflation expectation for the median income group.

Connection with existing survey evidence The predictions of our model are also broadly consistent with existing survey evidence on expectations. Due to dispersed, noisy information, the consensus forecast underreacts to new information, implying a positive correlation between forecast errors and forecast revisions, as documented in Coibion and Gorodnichenko (2015) using the SPF. At the individual level, this correlation would necessarily be equal to zero under rational expectations. Bordalo, Gennaioli, Ma, and Shleifer (2020) show that, in the SPF, forecasters tend to overreact to their signals, resulting in a negative correlation at the individual level.[25] The left panel of Figure 4.3

[25]To detect under-reaction at the consensus level and over-reaction at the individual level, we run group-specific CG and BGMS regressions of forecast errors on forecast revisions:

$$\text{CG}: \quad \pi_{t+1} - \bar{\mathbb{E}}_t[\pi_{t+1}] = \beta_{\text{CG}}\left(\bar{\mathbb{E}}_t[\pi_{t+1}] - \bar{\mathbb{E}}_{t-1}[\pi_{t+1}]\right) + \epsilon_{t+1},$$
$$\text{BGMS}: \quad \pi_{t+1} - \mathbb{E}_{it}[\pi_{t+1}] = \beta_{\text{BGMS}}\left(\mathbb{E}_{it}[\pi_{t+1}] - \mathbb{E}_{it-1}[\pi_{t+1}]\right) + \epsilon_{it+1},$$

displays our model's prediction for these two coefficients across different income groups, which are consistent with the existing evidence. Notably, the CG coefficient increases with household income while the BGMS coefficient decreases in magnitude, which are the footprints of ambiguity aversion. In Appendix H.2, we discuss the empirical counterparts of the CG and BGMS regression coefficients in the MSC and the SCE. The direction of change in the coefficients, as a function of income, is broadly consistent with the model predictions though the magnitudes are not directly comparable.[26]

The right panel shows the response of the average inflation forecast for the median income group to an inflationary shock over time. Due to the additional overreaction, associated with $1 + r_g$, the average forecast actually overshoots the true inflation rate. The implied sign-switching pattern is consistent with the empirical findings in Angeletos, Huo, and Sastry (2021).[27]

Robustness and alternatives In our model, we focus on the effect of inflation on labor income, thereby abstracting from the balance-sheet effects which Doepke and Schneider (2006) have documented could be more detrimental to richer households. In Appendix H.3, we argue, using data from the Survey of Consumer Finances, that these effects are unlikely to dominate the labor-income effects, especially for the bottom four quintiles of the income distribution, for whom capital and business income account for less than 7% of their total income. The claim that poorer households are more affected by inflation is also supported by the findings in Cao, Meh, Ríos-Rull, and Terajima (2021), who compute the welfare effects of inflation across the distribution in a quantitative general equilibrium model. We also document, using the Census Bureau's House Purse Survey, that poorer households consistently report higher levels of inflation concern. This is particularly relevant to our analysis, as the mechanism hinges directly on households' subjective perceptions of inflation. These last points are also reassuring against potential general equilibrium effects overturning the direction of inflation effects.[28]

The CAAA preferences used above are non-homothetic, and our mechanism relies on this to generate the disproportionate effect of inflation on poorer households (a notion similar to Cerreia-Vioglio, Maccheroni, and Marinacci (2022)). Alternatively, we could assume that households have non-homothetic preferences toward risk, due to borrowing limits or minimum consumption requirements. Then, homothetic preferences toward ambiguity would yield similar model predictions. Our assumptions are designed to align with our general theoretical framework, but this should be mainly seen

using artificial datasets simulated by our model.

[26]Due to limitations of the MSC and SCE data we cannot exactly construct the term structure of the forecasts. In Appendix H.2, we provide an approximate version of these regression coefficients.

[27]Note that in the IRF, we do not include the bias term. When conducting the projection method in Angeletos, Huo, and Sastry (2021), such bias terms are absorbed by the constant regressor.

[28]Our mechanism hinges on households' perceived joint distribution of aggregate variables, which may diverge from economic theory. As shown by Candia, Coibion, and Gorodnichenko (2020) and Han (2023), households typically associate higher inflation with lower real GDP growth. Our model reflects this by linking higher inflation with reduced real income, which broadly aligns with households' perceptions of inflation.

as a convenient modeling device.

There are also reasonable alternative explanations for at least part of the evidence described above. For instance, Broer, Kohlhas, Mitman, and Schlafmann (2022) show that the relationship between sensitivity to information and income levels can result from rational inattention, though this logic cannot explain the observed average bias. Another possibility is that households effectively report forecasts about the inflation rates of their own consumption baskets, and that individual inflation rates differ by households' income levels. However, as shown in Kaplan and Schulhofer-Wohl (2017), the annual inflation difference between the top and bottom income groups is about 1%, which can only account for half of the difference in bias, while the persistence of inflation rates across households' income is virtually identical.[29] Yet another explanation is that individual forecasts are subject to exogenous shocks. This could help explain the observed pattern between forecast persistence and income levels, for instance, but only if these exogenous shocks are sufficiently correlated across individuals within income groups and are more volatile for poorer households. Overall, we think our theory and these alternatives are complementary, and they jointly depict a more complete picture of the expectation formation process. Our preference for the ambiguity aversion channel is due to its ability to simultaneously speak to the patterns of bias and persistence in a parsimonious framework.

5 Application: Optimal Policy under Ambiguity

When agents in the private sector are ambiguity-averse, how should policy respond to changes in fundamentals, and how does additional sensitivity and bias affect policy design? In this section, we explore a policy game that builds on Barro and Gordon (1983) to shed light on these questions.

5.1 Environment

Time-consistent policy rule The policymaker chooses the inflation rate, π, so as to minimize the following social-loss function

$$\mathcal{L} = \mathbb{E}[U^2 + \omega(\pi - \pi^*)^2],$$

where U is the unemployment rate, π is the endogenous inflation rate, and π^* is an exogenous random inflation target that is drawn according to $\pi^* \sim \mathcal{N}(\overline{\pi}, \sigma_\pi^2)$.[30] The parameter ω balances the preference for lower unemployment against smaller deviations from the inflation target.

The policymaker faces a static Phillips curve that specifies how the unemployment rate reacts to average inflation surprises,

$$U = -\beta(\pi - \overline{\mathcal{F}}[\pi]),$$

[29] See Table 3 in the main text and Figure 8 in the online appendix of Kaplan and Schulhofer-Wohl (2017).

[30] We normalize $\overline{\pi} = 0$ later on, but the main results do not hinge on this normalization.

where β is the slope of the Phillips curve. The unemployment rate is lower when actual inflation exceeds the average expected inflation. Importantly, agents in the private sector may not be rational, and the expectation operator corresponds to agents' subjective expectations. We have implicitly normalized the natural unemployment rate to zero, $U^* = 0$. Therefore, if $\pi = \overline{\mathcal{F}}[\pi] = \pi^*$, the policymaker would achieve the first-best outcome.

We consider a discretionary scenario in which the policymaker cannot commit to a policy ex ante. As in Barro and Gordon (1983), the time-consistent inflation policy is given by

$$\pi = (1-\alpha)\,\pi^* + \alpha\,\overline{\mathcal{F}}[\pi], \quad \text{with} \quad \alpha \equiv \frac{\beta^2}{\omega + \beta^2}. \tag{5.1}$$

That is, the inflation rate is a weighted average between the exogenous inflation target and the economy-wide subjective inflation expectation.

Subjective expectations The exogenous inflation target cannot be perfectly observed by the public, and each agent i receives a private, noisy signal about it:

$$x_i = \pi^* + \epsilon_i, \quad \text{with} \quad \epsilon_i \sim \mathcal{N}(0, \sigma_\epsilon^2).$$

The noise here captures the notion that agents are inattentive to signals due to either attention costs or cognitive constraints. In addition to the informational frictions, agents in the private sector perceive ambiguity about the exogenous inflation target π^*. More specifically, agents believe that the mean of the target is ambiguous, that is,

$$\pi^* \sim \mathcal{N}(\mu, \sigma_\pi^2), \quad \text{and} \quad \mu \sim \mathcal{N}(\overline{\pi}, \sigma_\mu^2),$$

where σ_μ controls the amount of ambiguity.

The payoff of an agent depends on their subjective expectation and on inflation itself,

$$u(\mathcal{F}_i[\pi], \pi) = -(\mathcal{F}_i[\pi] - \pi)^2 - \chi\pi.$$

That is, agents care about the accuracy of their forecast, and an increase in the inflation rate directly reduces their utility. This utility function can be seen as a reduced-form version of the micro-founded structure introduced in Section 4,[31] which allows for biased subjective beliefs, in line with the data. In contrast with Section 4, inflation here is an endogenous equilibrium object that depends on the average subjective inflation expectation. As a result, the coordination motive is at play in shaping the subjective expectations.

[31] For simplicity, we abstract from the complications due to the second-order term and effectively assume households are homogeneous except for the dispersed information.

5.2 Optimal inflation policy

In equilibrium, the optimal policy that satisfies condition (5.1) can be characterized by a pair of policy parameters, $\mathcal{R}$ and $\mathcal{C}$, such that

$$\pi = \mathcal{R}\pi^* + \mathcal{C},$$

where $\mathcal{R}$ represents the responsiveness to the inflation target, and $\mathcal{C}$ determines the average level of inflation. How do aggregate inflation forecasts affect the optimal inflation policy?

We start with a rational expectations benchmark with informational frictions but in which agents do not perceive ambiguity.

Proposition 5.1. *With rational expectations ($\sigma_\mu^2 = 0$), the optimal policy rule is given by*

$$\mathcal{R}^{RE} = 1 - \alpha + \alpha \frac{\sigma_\pi^2}{\sigma_\pi^2 + (1-\alpha)^{-1}\sigma_\epsilon^2} \leq 1, \quad \textit{and} \quad \mathcal{C}^{RE} = 0.$$

Without ambiguity, the discretionary policy (5.1) can be viewed as a beauty contest game. When individual agents form expectations, they need to forecast the forecast of others, and the strength of these considerations is regulated by α. In the end, the average expectation is given by

$$\overline{\mathbb{E}}[\pi] = \frac{\sigma_\pi^2}{\sigma_\pi^2 + (1-\alpha)^{-1}\sigma_\epsilon^2}\,\pi^*,$$

where $(1-\alpha)^{-1}$ captures the discounting of private signals due to the coordination motive. Effectively, agents form their expectations using (w, α)-modified signals as in Section 3 with $w = 0$. Although the responsiveness is dampened due to dispersed information, the inflation rate remains proportional to the target, and $\mathcal{C}^{\mathrm{RE}} = 0$. With full information rational expectations (FIRE), the dampening effect vanishes, and the endogenous inflation perfectly tracks the target: $\mathcal{R}^{\mathrm{FIRE}} = 1$.

Next, consider the case in which agents are ambiguity-averse.

Proposition 5.2. *With ambiguity aversion, the aggregate subjective expectation is given by*

$$\overline{\mathcal{F}}[\pi] = \mathcal{S}\pi^* + \mathcal{B},$$

where

$$\mathcal{S} = \frac{(1+w)\,\sigma_\pi^2}{(1+w)\,\sigma_\pi^2 + (1-\alpha)^{-1}\sigma_\epsilon^2}, \qquad \mathcal{B} = \chi\lambda\,(1-\alpha+\alpha\mathcal{S})\,(1-\mathcal{S})\,\sigma_\mu^2,$$

and

$$w = \frac{\sigma_\mu^2/\sigma_\pi^2}{1 - 2\lambda\,(1-\alpha)^2\,(1-\mathcal{S})^2\,\sigma_\mu^2}.$$

The implied inflation policy satisfies

$$\mathcal{R} = 1 - \alpha + \alpha\mathcal{S} \in [\mathcal{R}^{RE}, 1], \quad and \quad \mathcal{C} = \alpha\mathcal{B}.$$

Proposition 5.2 echoes our earlier emphasis on the effects of ambiguity aversion: the sensitivity to signals is amplified through the endogenously perceived prior, captured by the factor $(1 + w)$, and the expectation is permanently biased upwards by $\mathcal{B}$.

The coordination motive, controlled by α, affects the equilibrium outcome in two ways. First, the policymaker internalizes the behavioral patterns of the private sector and adjusts the actual inflation process accordingly. This leads to higher responsiveness, $\mathcal{R} \geq \mathcal{R}^{\text{RE}}$, and a lifted intercept, $\mathcal{C} > 0$. The extent to which the inflation policy inherits this behavior from the private sector is mechanically increasing in the degree of coordination motive α. Second, the effect of ambiguity on an agent's subjective belief also hinges on the strength of the coordination motive, as both w and $\mathcal{B}$ depend on α directly. Different from the analysis in Section 3, the economy here is inefficient even with complete information, and the socially optimal coordination level à la Angeletos and Pavan (2007) is $\widehat{\alpha} \equiv 1 - (1 - \alpha)^2$, which is the key statistics that determines effects of ambiguity via w.[32]

Figure 5.1: Inflation Policy with and without Ambiguity

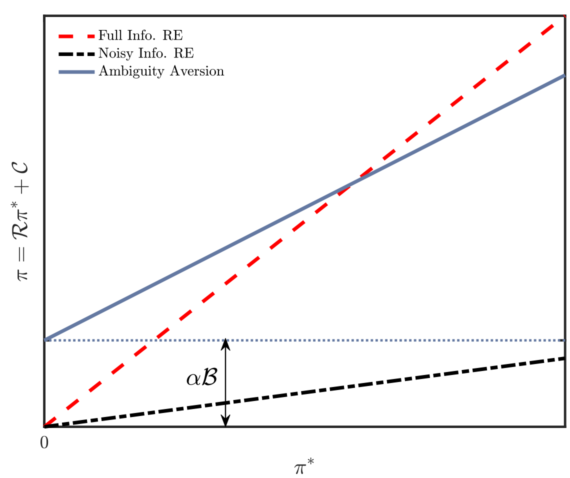


Figure 5.1 depicts the increased slope and shifted intercept that result from introducing ambiguity aversion. The red-dashed line represents the inflation policy under FIRE, which aligns exactly with the 45-degree line. The black-broken line displays the policy rule with noisy information but without ambiguity. Relative to FIRE, the only change is the dampened responsiveness. In both cases, there is no bias. When agents are ambiguity-averse, we get the blue-solid line. The slope of the

[32]We provide a detailed analysis of inefficient economies in Appendix B.2.

policy approaches the one under FIRE, but this enhanced responsiveness is accompanied by a higher intercept due to the presence of bias in agents' forecasts.

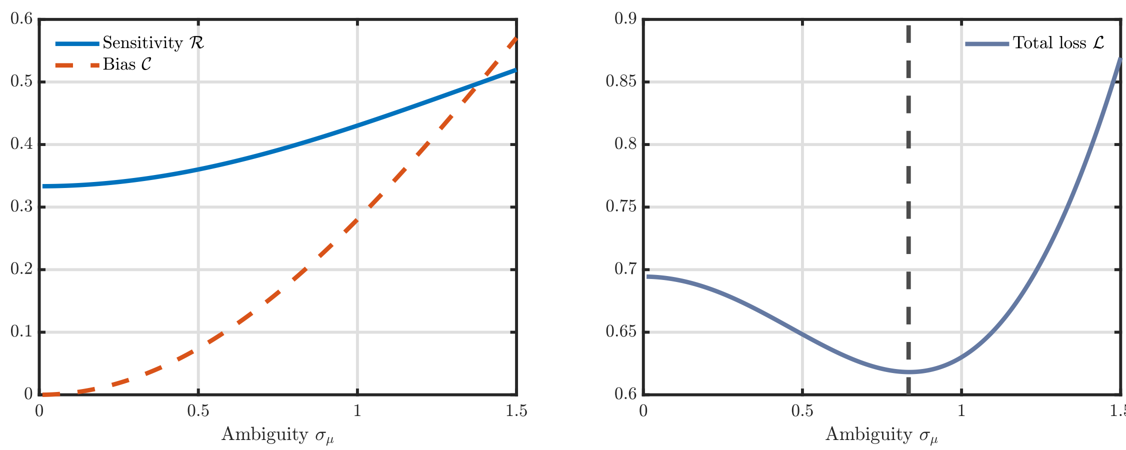
5.3 Ambiguity and social welfare

Given an inflation policy rule, social welfare losses can be expressed as follows:

$$\mathcal{L} = \mathbb{E}[U^2 + \omega(\pi - \pi^*)^2] = \frac{\omega}{\alpha}\left[(1-\mathcal{R})^2\sigma_\pi^2 + \mathcal{C}^2\right].$$

The inflation policy under FIRE, with $\mathcal{R} = 1$ and $\mathcal{C} = 0$, achieves the first best. Further, either an increase in responsiveness, towards $\mathcal{R} = 1$, or a reduction in the magnitude of the intercept, $\mathcal{C}$, would improve welfare. Intuitively, the policymaker would like the public to pay attention to the inflation target, facilitating the implementation of the desired inflation level. Any irrelevant deviation of the average expectation from π^* is socially inefficient.

FIGURE 5.2: Bias, Sensitivity, and Social Welfare



(A) Bias and sensitivity (B) Social loss $\mathcal{L}$

When information is noisy, the responsiveness of the private sector is dampened and the introduction of ambiguity can help approach the policy under FIRE. As can be observed in Figure 5.1, the combined effect of higher responsiveness and higher bias can lead to a better approximation of the FIRE first-best policy. This result is formalized in the following proposition.

Proposition 5.3. *Fixing the amount of noise σ_ϵ^2, there exists an intermediate level of ambiguity $\sigma_\mu^2 > 0$ that minimizes the social loss $\mathcal{L}$.*

Figure 5.2 helps illustrate the basic idea. The left panel shows how the responsiveness, $\mathcal{R}$, and the intercept, $\mathcal{C}$, vary with the amount of ambiguity, σ_μ. On one hand, the introduction of ambiguity enhances the responsiveness to signals, as the private sector shifts attention from their ambiguous priors to their signals, which contain relevant information about the policy target. On the other hand, it generates bias, as the private sector becomes increasingly concerned about the realization of a high-inflation model. These two competing effects generate a U-shape social loss function, which is minimized at an intermediate level of ambiguity.

This mechanism could justify, for instance, the recent switch by the Fed from a fixed inflation target to average inflation targeting (AIT). Jia and Wu (2023) argue that the adoption of AIT by the Fed in 2020 introduced ambiguity about the target to the extent that the time horizon used to compute the average is unclear.[33]

Alternative type of bounded rationality So far, we have rationalized the bias in agents' forecasts with ambiguity aversion. However, other types of bounded rationality could lead to the observed bias and this could lead to different policy implications.

As an example, consider a heterogeneous prior approach à la Angeletos, Collard, and Dellas (2018). Specifically, suppose that each agent i understands that $\pi^* \sim \mathcal{N}(\overline{\pi}, \sigma_\pi^2)$, but believes that all other agents perceive the inflation target with a bias, $\pi^* \sim \mathcal{N}(\overline{\pi} + \mathcal{B}, \sigma_\pi^2)$. Further, suppose all agents continue to receive a noisy signal about the inflation target. In this case, without ambiguity, the optimal inflation policy also features a non-zero intercept, but the welfare implication is significantly different.

Proposition 5.4. *With heterogeneous priors, the inflation policy rule is given by*

$$\mathcal{R} = \mathcal{R}^{RE}, \quad \textit{and} \quad \mathcal{C} = \alpha \left(\mathcal{R}^{RE} - 1 \right) \mathcal{B}.$$

The social loss monotonically increases with $\mathcal{B}$.

In contrast to the results derived above, if the observed bias in inflation forecasts is attributed to heterogeneous priors, a higher bias necessarily implies lower welfare. Since there is no concomitant effect on the sensitivity to signals, bias simply leads to less accurate average expectations.

6 Connection with Robust Preference

In this section, we document an intimate connection between the smooth model of ambiguity and the robust preferences model (Hansen and Sargent, 2001a,b). Although these two approaches to model uncertainty are conceptually different,[34] the main theoretical insights developed earlier about

[33] Jia and Wu (2023) also show that the ambiguity generated by this switch can be beneficial as it allows the Fed to increase its credibility. This benefit of ambiguity is different from the one we highlight. Their setup abstracts from ambiguity aversion, which plays a crucial role in our results.

[34] We refer to Hansen and Marinacci (2016) for a comprehensive discussion.

sensitivity and bias, along with the observational equivalence to Bayesian forecasts, also apply to models with robust preferences.

In parallel with the smooth model of ambiguity, we consider an efficient economy in which the utility function is given by

$$u(k_{it}, K_t, \xi_t) = -\frac{1}{2}\left[(1-\alpha)\left(k_{it}-\xi_t\right)^2 + \alpha\left(k_{it}-K_t\right)^2\right] - \chi\xi_t - \frac{1}{2}\gamma\xi_t^2,$$

and the signals follow the same general processes described in Section 3.1.

We model robust preferences following Hansen and Sargent (2005). Agents worry about potential model misspecification and consider a set of alternatives:

$$\max_{k_{it}} \min_{m_{it}} \quad \mathbb{E}_{it}\left[u(k_{it}, K_t, \xi_t)\, m_{it} + \frac{1}{\varpi} m_{it} \log m_{it}\right] \tag{6.1}$$

$$\text{s.t.} \quad m_{it} > 0, \quad \text{and} \quad \mathbb{E}_{it}\left[m_{it}\right] = 1.$$

Here, each random variable m_{it} introduces a distorted distribution, generating an alternative model, where $\mathbb{E}_{it}[m_{it} \log m_{it}]$ corresponds to the relative entropy. The parameter ϖ controls the extent to which agents desire robustness. Agents then choose their strategies to optimize the worst-case scenario across the set of models under consideration.

Just as in the smooth model of ambiguity, under robust preferences, the subjective expectations and strategies of agents are jointly determined. Despite these complex interactions, the equilibrium strategy ultimately takes a simple form similar to the one in the smooth model.

Proposition 6.1. *The linear strategy under robust preferences takes the following form*

$$g\left(x_i^t\right) = (1+r)\, p\left(L; w, \alpha\right) x_{it} + \mathcal{B}.$$

1. *The polynomial matrix $p(L; w, \alpha)$ is the Bayesian forecasting rule with the (w, α)-modified signal process and w satisfies*

$$w = \frac{\varkappa_2}{1-\alpha}; \tag{6.2}$$

2. *The additional amplification, r, satisfies*

$$r = \frac{\varkappa_1 - \varkappa_2}{1-\alpha+\varkappa_2}; \tag{6.3}$$

3. *The level of bias, $\mathcal{B}$, satisfies*

$$\mathcal{B} = \chi\frac{r}{\gamma}. \tag{6.4}$$

The two endogenous scalars $(\varkappa_1, \varkappa_2)$ are such that

$$\varkappa_1 - \varkappa_2 = \varpi\gamma\,(1-\alpha+\varkappa_1)\,\mathbb{V}_{it}(\xi_t) + \varpi\gamma\,(\alpha-\varkappa_2)\,\mathbb{COV}_{it}(K_t,\xi_t),$$
$$\varkappa_2 = \frac{\alpha\,(1-\alpha)}{\gamma}\,(\varkappa_1-\varkappa_2) - \varpi\alpha\,(1-\alpha)\,\mathbb{DISP}(k_{it}),$$

where $\mathbb{V}_{it}(\xi_t)$, $\mathbb{COV}_{it}(K_t,\xi_t)$, and $\mathbb{DISP}(k_{it})$ denote the conditional volatility of the fundamental, the conditional covariance between the aggregate action and the fundamental, and the unconditional cross-sectional dispersion of individual actions, respectively.

Under robust preferences, agents want their actions to be robust across various models. In contrast, the smooth model of ambiguity is isomorphic to a setup where agents look for robustness across various priors about a fixed set of possible models (Hansen and Marinacci, 2016). Despite the conceptual differences between these two approaches, Proposition 6.1 shows that they are, in a sense, observationally equivalent. The following corollary in turn provides the condition under which the two models are equivalent in terms of individual strategies, with only a difference in the amount of bias.

Corollary 6.1. *Fix the objective environment. For a robust preferences model that satisfies*

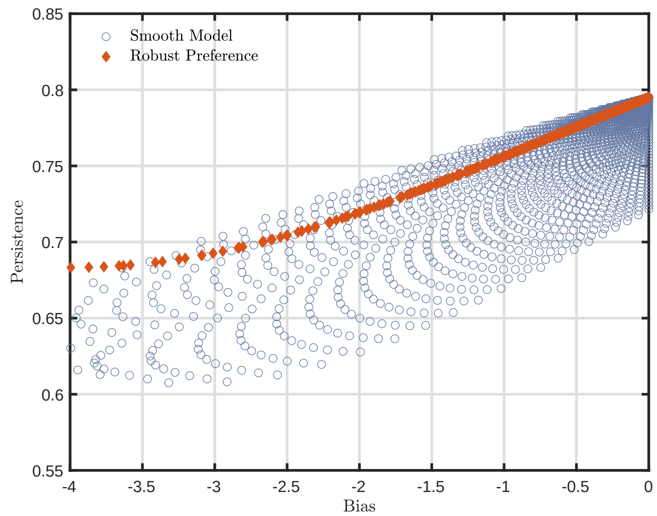
1. $w \geq 0, r \geq 0, \mathcal{S} \leq 1$,
2. $(1-\mathcal{S})\left(\frac{\gamma w}{(1+w)r} - \frac{(1-\alpha)(1+w)r}{w}\right) + \gamma > (1-\alpha)\frac{\mathbb{V}(\xi_t - K_t)}{\mathbb{V}(\xi_t)}$,

there exists a smooth model of ambiguity such that the equilibrium strategies in these two models are identical up to a constant.

Although the equilibrium strategies share a similar transformation in terms of Bayesian expectations, there are still some subtle differences between the two approaches. In the smooth model of ambiguity, we focus on the case where the model uncertainty is only about the prior mean of the fundamental. With robust preferences, the model uncertainty is about the entire stochasticity of the environment. When determining the endogenous amplification parameter w, the former only requires unconditional moments of aggregate variables, while the latter requires conditional moments of both aggregate and individual variables.

Another difference between the two approaches is that the smooth model allows a separation between the attitude towards ambiguity, λ, and the amount of ambiguity, σ_μ, whereas the robust preferences model is more parsimonious, with a single parameter, ϖ, to control the overall concern about model misspecification. To demonstrate the quantitative difference between the two approaches, we revisit the inflation-expectations application. Specifically, we keep the objective environment fixed,

Figure 6.1: A Comparison between SM and RP



Note: This figure compares the predictions on the bias and persistence of forecast errors between the smooth model and the robust preferences model. Each blue circle corresponds a smooth model with a choice of (λ, σ_μ^2). Each red dot corresponds to a robust preferences model with a choice of ϖ. The objective environment is fixed the same as that in Section 4.

including the exogenous stochastic process of inflation, ρ and σ_η, the private signal noise, σ_ϵ, and the preference parameters, β and ν. We examine the bias and persistence of forecast errors for households with median income. For the smooth model, we vary both the degree of ambiguity aversion, λ, and the amount of ambiguity, σ_μ^2. For the robust preferences model, we vary the parameter controlling the preference for robustness, ϖ.

Figure 6.1 presents the joint distribution of bias and persistence of forecast errors. Each blue circle corresponds to the outcome under a specific pair of (λ, σ_μ^2) in the smooth model, while each red diamond point represents the outcome under a particular choice of ϖ in the robust preferences model. Both approaches yield the same qualitative prediction: higher bias is associated with lower persistence. Moreover, due to the additional degree of freedom in the smooth model, the scatter plot covers the line implied by the robust preferences model. Quantitatively, the two approaches also produce comparable predictions.

7 Conclusion

In this paper, we study the effects of ambiguity in a general equilibrium environment with incomplete information. We provide an equivalence result that characterizes the equilibrium strategy as the solution to an adjusted single-agent Bayesian forecasting problem. Ambiguity aversion induces additional sensitivity to signals and a pessimistic bias, with both effects depending endogenously on agents' payoffs and the market structure. These properties allow us to match salient patterns observed in survey evidence on expectations, which are difficult to explain with rational expectation models. We also show that additional sensitivity and bias in subjective beliefs can change the effect of policy instruments, and that the exact micro-foundation for such belief distortions matters for policy design.

While our focus thus far has been on payoff structures that are homogeneous across agents, it would be interesting to explore the extent to which our main insights can be extended to network games. For example, firms in production supply chains may have concerns about their upstream suppliers and downstream customers perceiving different models. Another direction for future research is to explore how policymakers should incorporate substantial deviations of subjective beliefs from rational ones into the design of monetary and fiscal policies in quantitative models.

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Online Appendix for "Bias and Sensitivity under Ambiguity"

Zhen Huo Marcelo Pedroni Guangyu Pei

A Proofs of Main Results

In this appendix, we present the proofs of the main results from Section 3. We start by proving Proposition 3.2, which yields the fixed point conditions that characterize the equilibrium. We proceed by proving the general equivalence result, Proposition 3.3, based on which we can prove the existence of equilibrium, Proposition 3.1, as well as the comparative statics of sensitivity $\mathcal{S}$ and bias $\mathcal{B}$ with respect to the coordination motive α, Proposition 3.5.

Proof of Proposition 3.2. The equilibrium concept from Definition 3.1 is equivalent to the notion of ex-ante equilibrium from Hanany, Klibanoff, and Mukerji (2020). It is equivalent to the characterization of sequential equilibria with ambiguity (SEA) when conditional preferences are updated using the smooth rule of updating proposed in Hanany and Klibanoff (2009). The key for the equilibrium refinement of SEA is to ensure dynamic consistency, in the sense that ex-ante contingent plans are respected ex-post with the arrival of new information. Specifically, conditional on the realization of any possible history of private information, x_i^t, the optimal strategy of agent i maximizes their conditional preference, given by

$$\phi^{-1}\left(\int_{\mu^t}\phi\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it},K_t,\xi_t\right)\mid x_i^t\right]\right)\tilde{p}\left(\mu^t\mid x_i^t\right)\mathrm{d}\mu^t\right), \tag{A.1}$$

where $\mathbb{E}^{\mu^t}\left[u\left(k_{it},K_t,\xi_t\right)\mid x_i^t\right]$ denotes the expected utility conditional on x_i^t under a particular model μ^t. The interim belief system is characterized by a posterior belief $\tilde{p}\left(\mu^t\mid x_i^t\right)$ that follows the smooth rule of updating:

$$\tilde{p}\left(\mu^t\mid x_i^t\right)\propto\underbrace{\frac{\phi'\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it}^*,K_t^*,\xi_t\right)\right]\right)}{\phi'\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it}^*,K_t^*,\xi_t\right)\mid x_i^t\right]\right)}}_{\text{Weights}}\underbrace{p\left(x_i^t\mid\mu^t\right)p\left(\mu^t\right)}_{\text{Bayesian Kernel}},$$

where $\{k_{it}^*\left(x_i^t\right)\}_{x_i^t,i}$ denotes the equilibrium strategy profiles in the cross-section of the economy and $K_t^*\equiv\int_i k_{it}^*\mathrm{d}i$ denotes the equilibrium aggregate action.

The first-order condition of maximizing (A.1) with respect to k_{it} yields

$$\int_{\mu^t}\phi'\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it},K_t,\xi_t\right)\mid x_i^t\right]\right)\frac{\partial\mathbb{E}^{\mu^t}\left[u\left(k_{it},K_t,\xi_t\right)\mid x_i^t\right]}{\partial k_{it}}\tilde{p}\left(\mu^t\mid x_i^t\right)\mathrm{d}\mu^t=0.$$

Since

$$\frac{\partial\mathbb{E}^{\mu^t}\left[u\left(k_{it},K_t,\xi_t\right)\mid x_i^t\right]}{\partial k_{it}}=k_{it}-\left(1-\alpha\right)\mathbb{E}^{\mu^t}\left[\xi_t\mid x_i^t\right]-\alpha\mathbb{E}^{\mu^t}\left[K_t\mid x_i^t\right],$$

the first-order condition can be used to solve for the optimal strategies $\{k_{it}^*\left(x_i^t\right)\}_{x_i^t,i}$,

$$k_{it}^*(x_i^t)=\int_{\mu^t}\left(\left(1-\alpha\right)\mathbb{E}^{\mu^t}\left[\xi_t\mid x_i^t\right]+\alpha\mathbb{E}^{\mu^t}\left[K_t^*\mid x_i^t\right]\right)\hat{p}\left(\mu^t\mid x_i^t\right)\mathrm{d}\mu^t,$$

with

$$\hat{p}\left(\mu^t\mid x_i^t\right)\equiv\frac{\phi'\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it}^*,K_t^*,\xi_t\right)\right]\right)p\left(x_i^t\mid\mu^t\right)p\left(\mu^t\right)}{\int_{\mu^t}\phi'\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it}^*,K_t^*,\xi_t\right)\right]\right)p\left(x_i^t\mid\mu^t\right)p\left(\mu^t\right)\mathrm{d}\mu^t},$$

which completes the proof. $\square$

Proof of Proposition 3.3. Following Huo and Pedroni (2020), we first consider a truncated version of our model. After solving this truncated version, the appropriate limits yield the desired result.[35]

Fix t and define

$$\vartheta \equiv \xi_t = \sum_{k=0}^{\infty} a_k \eta_{t-k}.$$

Let ϑ_q denote the MA(q) truncation of ϑ, such that

$$\vartheta_q = \sum_{k=0}^{q} a_k \eta_{t-k},$$

and let $x_{p,i}^N \equiv \{x_{p,it}, \cdots, x_{p,it-N}\}$, with $x_{p,it-k}$ denoting the MA(p) truncation of x_{it-k}.

Consider the truncated problem of forecasting the the fundamental ϑ_q given $x_{p,i}^N$. To further ease notation, define

$$\eta \equiv \begin{bmatrix} \eta_t \\ \vdots \\ \eta_{t-T} \end{bmatrix}, \quad \mu \equiv \begin{bmatrix} \mu_t \\ \vdots \\ \mu_{t-T} \end{bmatrix}, \quad \epsilon_i \equiv \begin{bmatrix} \epsilon_{it} \\ \vdots \\ \epsilon_{it-T} \end{bmatrix}, \quad \text{and} \quad \nu_i \equiv \begin{bmatrix} \eta \\ \epsilon_i \end{bmatrix}$$

Let R denote the length of $x_{p,i}^N$, and N the length of ϵ_{it}. It follows that, there exists a vector a with length $u \equiv T+1$, and a matrix B with dimensions $n \times m$, where $n \equiv R(T+1)$ and $m \equiv (1+N)(T+1)$, such that the truncated fundamental and the private signals are given by

$$\theta \equiv \vartheta_q = A\nu_i, \quad \text{with} \quad A \equiv \left[a', 0'_{m-u,1}\right], \quad \text{and} \quad x_i \equiv x_{p,i}^N = B\nu_i,$$

where $0_{m-u,1}$ is an $(m-u) \times 1$ vector of zeros. In the objective environment, ν_i is normally distributed,

$$\nu_i \sim \mathcal{N}(0, \Omega), \quad \text{with} \quad \Omega = \begin{bmatrix} \sigma_\eta^2 \mathrm{I}_u & 0 \\ 0 & \Xi \end{bmatrix},$$

where I_u denotes the identity matrix of size u and Ξ denotes the variance-covariance matrix of the $(m-u) \times 1$ vector of idiosyncratic shocks, ϵ_i. Subjectively, agents believe that η is drawn from a Gaussian distribution with variance-covariance matrix $\sigma_\eta^2 \mathrm{I}_u$ but there is uncertainty about its prior mean, denoted by μ. Ambiguity is then captured by the perception that

$$\eta \sim \mathcal{N}\left(\mu, \sigma_\eta^2 \mathrm{I}_u\right), \quad \text{and} \quad \mu \sim \mathcal{N}(0, \Omega_\mu), \quad \text{with} \quad \Omega_\mu \equiv \sigma_u^2 \mathrm{I}_u.$$

From Proposition 3.2, we know that the best response of agent i satisfies

$$k_i = \int_\mu \big((1-\alpha)\mathbb{E}^\mu[\theta \mid x_i] + \alpha \mathbb{E}^\mu[K \mid x_i]\big)\hat{p}(\mu \mid x_i)\,\mathrm{d}\mu, \tag{A.2}$$

[35] See Online Appendix A.1 of Huo and Pedroni (2020) for detailed proofs.

with

$$\hat{p}(\mu \mid x_i) \propto \exp\left(-\lambda \mathbb{E}^{\mu}\left[u(k_i, K, \theta)\right]\right) p(x_i \mid \mu) p(\mu).$$

We proceed by using a guess-and-verify strategy. First, we guess a symmetric linear equilibrium that

$$k_i = h' B \nu_i + h_0 \qquad \forall i.$$

We can show that ex-ante expected utility, under a particular model μ, is such that

$$\begin{aligned}
\mathbb{E}^{\mu}\left[u(k_i, K, \theta)\right] = & -\mu' \left[\frac{1}{2}(1-\alpha)\mathcal{K}(A' - B'h)(A - h'B)\mathcal{K}' + \frac{1}{2}\gamma \mathcal{K} A' A \mathcal{K}'\right]\mu \\
& + \left[\frac{1}{2}(1-\alpha) h_0 (A - h'B)\mathcal{K}' + \frac{1}{2}\chi A \mathcal{K}'\right]\mu + \mu' \left[\frac{1}{2}(1-\alpha) h_0 \mathcal{K}(A' - B'h) + \frac{1}{2}\chi \mathcal{K} A'\right] \\
& \underbrace{-\frac{1}{2}(1-\alpha)(A - h'B)\Omega(A' - B'h) - \frac{1}{2}(1-\alpha) h_0^2 - \frac{1}{2}\alpha h' B (\mathrm{I}_m - \Lambda)\Omega B' h - \frac{1}{2}\gamma A \Omega A,}_{\text{independent of } \mu}
\end{aligned} \tag{A.3}$$

where matrices $\mathcal{K}$ and Λ are such that

$$\mathcal{K} \equiv \left[\mathrm{I}_u, 0_{u,m-u}\right], \quad \text{and} \quad \Lambda \equiv \mathcal{K}'\mathcal{K}.$$

At the same time, we have that

$$p(\mu \mid x_i) \propto \exp\left(-\frac{1}{2}\mu' \left(\mathcal{K} B' (B\Omega B')^{-1} B \mathcal{K}'\right)^{-1} \mu + \frac{1}{2}\mu' \mathcal{K} (B\Omega B')^{-1} x_i + \frac{1}{2} x_i' (B\Omega B')^{-1} \mathcal{K}' \mu\right).$$

It follows that

$$\hat{p}(\mu \mid x_i) \propto \exp\left(-\frac{1}{2}\mu' S^{-1}\mu + \frac{1}{2}\mu' S^{-1}(M x_i + \pi) + \frac{1}{2}(M x_i + \pi)' S^{-1}\mu\right),$$

where matrices M, π, and S are such that

$$M \equiv S\mathcal{K}(B\Omega B')^{-1}, \quad \pi \equiv S\left[-\lambda(1-\alpha) h_0 \mathcal{K}(A' - B'h) + \lambda \chi \mathcal{K} A'\right],$$

and

$$S \equiv \left(\mathcal{K} B' (B\Omega B')^{-1} B \mathcal{K}' + \Omega_{\mu}^{-1} - \lambda\left[(1-\alpha)\mathcal{K}(A' - B'h)(A - h'B)\mathcal{K}' + \gamma \mathcal{K} A' A \mathcal{K}'\right]\right)^{-1}.$$

Accordingly, we can show that the subjective expectations are such that

$$\int_{\mu} \mathbb{E}^{\mu}\left[\theta \mid x_i\right] \hat{p}(\mu \mid x_i)\, \mathrm{d}\mu = \mathrm{T}\, x_i + (A - \mathrm{T}B)\mathcal{K}' \left[S\mathcal{K}B'(B\Omega B')^{-1} x_i + \pi\right],$$

and

$$\int_{\mu} \mathbb{E}^{\mu}\left[K \mid x_i\right] \hat{p}(\mu \mid x_i)\, \mathrm{d}\mu = \mathrm{H}\, x_i + h'(B\Lambda - \mathrm{H}B)\mathcal{K}' \left[S\mathcal{K}B'(B\Omega B')^{-1} x_i + \pi\right] + h_0,$$

where matrices T and H are given by

$$\mathrm{T} \equiv A\Omega B' \left(B\Omega B'\right)^{-1}, \quad \text{and} \quad \mathrm{H} \equiv B\Lambda\Omega B' \left(B\Omega B'\right)^{-1}.$$

Therefore, matching coefficients leads to the following equilibrium conditions for h and h_0,

$$h' = (1-\alpha)\,\mathrm{T} + \alpha h'\mathrm{H} + \left[(1-\alpha)\,(A - \mathrm{T}B) + \alpha h'\,(B\Lambda - \mathrm{H}B)\right]\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}, \tag{A.4}$$

and

$$(1-\alpha)\,h_0 = \left[(1-\alpha)\,(A - \mathrm{T}B) + \alpha h'\,(B\Lambda - \mathrm{H}B)\right]\mathcal{K}'\pi. \tag{A.5}$$

In what follows, we first focus on equation (A.4). Through a sequence of lemmas, we show that this fixed-point problem for h can be recast as the solution of a pure forecasting problem. We then proceed to characterize h_0 using equation (A.5).

The next lemmas are organized as follows. Lemma A.1 rewrites the equilibrium condition for h described above as a beauty-contest problem with a modified variance-covariance matrix. Lemma A.2 establishes that h can be obtained by solving a forecasting problem with a modified variance-covariance matrix. Lemma A.3 simplifies the variance-covariance matrix of the forecasting problem, and Lemma A.4 further simplifies it yielding a symmetric variance-covariance matrix. After the lemmas we take the limits of the truncated forecasting problem as $T \to \infty$.

Lemma A.1. *Define*

$$\hat{\Omega} \equiv \Omega + \mathcal{K}'W\mathcal{K}, \quad \hat{\mathrm{T}} \equiv A\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}, \quad \hat{\mathrm{H}} \equiv B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1},$$

and

$$W \equiv \left(\Omega_\mu^{-1} - \lambda\left[(1-\alpha)\,\mathcal{K}\,(A' - B'h)\,(A - h'B)\,\mathcal{K}' + \gamma\mathcal{K}A'A\mathcal{K}'\right]\right)^{-1}.$$

Then, the equilibrium h solves the following fixed-point problem

$$h' = (1-\alpha)\,\hat{\mathrm{T}} + \alpha h'\hat{\mathrm{H}}.$$

Proof. Using the Woodbury matrix identity, we have that

$$\begin{aligned}\left(B\hat{\Omega}B'\right)^{-1} &= \left(B\Omega B' + B\mathcal{K}'W\mathcal{K}B'\right)^{-1}\\ &= \left(B\Omega B'\right)^{-1} - \left(B\Omega B'\right)^{-1}B\mathcal{K}'\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}' + W^{-1}\right)^{-1}\mathcal{K}B'\left(B\Omega B'\right)^{-1}\\ &= \left(B\Omega B'\right)^{-1} - \left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1},\end{aligned} \tag{A.6}$$

Then, if $\hat{h}$ is such that $\hat{h}' = (1-\alpha)\,\hat{\mathrm{T}} + \alpha\hat{h}'\hat{\mathrm{H}}$, we have that

$$\begin{aligned}
\hat{h}' =&\ (1-\alpha)\,A\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} + \alpha\hat{h}'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} \\
=&\ (1-\alpha)\,A\left(\Omega+\mathcal{K}'W\mathcal{K}\right)B'\left(B\hat{\Omega}B'\right)^{-1} + \alpha\hat{h}'B\Lambda\left(\Omega+\mathcal{K}W\mathcal{K}'\right)B'\left(B\hat{\Omega}B'\right)^{-1} \\
=&\ (1-\alpha)\,A\Omega B'\left(B\hat{\Omega}B'\right)^{-1} + (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\hat{\Omega}B'\right)^{-1} \\
&+ \alpha\hat{h}'B\Lambda\Omega B'\left(B\hat{\Omega}B'\right)^{-1} + \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\hat{\Omega}B'\right)^{-1}.
\end{aligned}$$

Using equation (A.6), it follows that

$$\begin{aligned}
\hat{h}' =&\ (1-\alpha)\,A\Omega B'\left(B\Omega B'\right)^{-1} - (1-\alpha)\,A\Omega B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&+ (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&+ \alpha\hat{h}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1} - \alpha\hat{h}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&+ \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
=&\ \underbrace{(1-\alpha)\,A\Omega B'\left(B\Omega B'\right)^{-1}}_{(1-\alpha)\mathrm{T}} + \underbrace{\alpha\hat{h}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1}}_{\alpha\hat{h}'\mathrm{H}} \\
&- \underbrace{(1-\alpha)\,A\Omega B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}}_{(1-\alpha)\mathrm{T}B\mathcal{K}'S\mathcal{K}B'(B\Omega B')^{-1}} - \underbrace{\alpha\hat{h}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}}_{\alpha\hat{h}'\mathrm{H}B\mathcal{K}'S\mathcal{K}B'(B\Omega B')^{-1}} \\
&+ (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&+ \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}.
\end{aligned}$$

Further, notice that the terms in the second-to-last line can be rewritten as

$$\begin{aligned}
&(1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
=&\ (1-\alpha)\,A\mathcal{K}'W\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'+W^{-1}\right)\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'+W^{-1}\right)^{-1}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&- (1-\alpha)\,A\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'+W^{-1}\right)^{-1}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
=&\ (1-\alpha)\,A\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1},
\end{aligned}$$

and, similarly, the terms in the last line can be rewritten as

$$\begin{aligned}
&\alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1} - \alpha\hat{h}'B\Lambda\mathcal{K}'W\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&= \alpha\hat{h}'B\Lambda\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}.
\end{aligned}$$

Therefore, we have that

$$\hat{h}' = (1-\alpha)\,\mathrm{T} + \alpha\hat{h}'\mathrm{H} + \left[(1-\alpha)\left(A-\mathrm{T}B\right) + \alpha\hat{h}'\left(B\Lambda-\mathrm{H}B\right)\right]\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1},$$

which is equivalent to the expression for h in equation (A.4). □

Lemma A.2. *Define*

$$\Omega_\Gamma \equiv \Gamma\hat{\Omega}, \quad \textit{with} \quad \Gamma \equiv \begin{bmatrix} \mathrm{I}_u & 0_{u,m-u} \\ 0_{m-u,u} & \frac{\mathrm{I}_{m-u}}{1-\alpha} \end{bmatrix}.$$

Then, the equilibrium h satisfies

$$h' = A\Omega_\Gamma B' \left(B\Omega_\Gamma B'\right)^{-1}.$$

Proof. Follows directly from Lemma A.1 and Theorem 1 in Huo and Pedroni (2020). □

Lemma A.3. *Define*

$$\Delta \equiv \Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}, \quad \textit{and} \quad \tilde{\Omega}_\mu \equiv \left(\Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1},$$

with the scalar $\hat{w}$ given by

$$\hat{w} \equiv \frac{\tau_\mu}{1 - \lambda(1-\alpha)\left(A - h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)}.$$

Then, the equilibrium h satisfies

$$h' = A\Delta B' \left(B\Delta B'\right)^{-1}.$$

Proof. It follows from Lemma A.2 that

$$\left(A - h'B\right)\Omega_\Gamma B' = 0,$$

and from the definition of Ω_Γ and $\tilde{\Omega}_\mu$ we have that

$$\Omega_\Gamma = \Gamma\Omega + \mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} - \lambda(1-\alpha)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}.$$

It is then sufficient to show that

$$\left(A - h'B\right)\left(\Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\right) = \left(A - h'B\right)\left(\Gamma\Omega + \mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} - \lambda\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}\right),$$

or, equivalently,

$$\begin{aligned}\hat{w}\tau_\mu^{-1}\left(A - h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K} &= \left(A - h'B\right)\mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} - \lambda\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)^{-1}\mathcal{K} \\ &= \left(A - h'B\right)\mathcal{K}'\left(\mathrm{I}_u - \lambda\left(1-\alpha\right)\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)^{-1}\tilde{\Omega}_\mu\mathcal{K}.\end{aligned}$$

Thus, it is sufficient to establish that

$$\hat{w}\tau_\mu^{-1}\left(A - h'B\right)\mathcal{K}' = \left(A - h'B\right)\mathcal{K}'\left(\mathrm{I}_u - \lambda\left(1-\alpha\right)\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)^{-1},$$

or

$$\hat{w}\tau_\mu^{-1}\left(A - h'B\right)\mathcal{K}'\left(\mathrm{I}_u - \lambda\left(1-\alpha\right)\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right) = \left(A - h'B\right)\mathcal{K}',$$

which can be rewritten as

$$\hat{w}\tau_\mu^{-1}\left(1-\lambda\left(1-\alpha\right)\left(A-h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(A'-B'h\right)\right)\left(A-h'B\right)\mathcal{K}'=\left(A-h'B\right)\mathcal{K}'.$$

The definition of $\hat{w}$ then yields the result. □

Lemma A.4. *Define*

$$\bar{\Delta}\equiv\Gamma\Omega+\hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu\mathcal{K},$$

with the scalar $\hat{w}$ given by

$$\hat{w}=\frac{\tau_\mu}{1-\lambda\left(1-\alpha\right)\left(A-h'B\right)\mathcal{K}'\left(\Omega_\mu+\frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\mathcal{K}\left(A'-B'h\right)}.$$

Also, let the scalar $\hat{r}$ be given by

$$\hat{r}\equiv\frac{\hat{w}}{1+\hat{w}}\left(\frac{\lambda\gamma}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\left(A-h'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A'.$$

Then, the equilibrium h satisfies

$$h'=(1+\hat{r})A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1}.$$

Proof. From the definition of $\tilde{\Omega}_\mu$ and Δ in Lemma A.3, we have that

$$\tilde{\Omega}_\mu\equiv\left(\Omega_\mu^{-1}-\lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1}=\Omega_\mu+\frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'},$$

and

$$\Delta\equiv\Gamma\Omega+\hat{w}\tau_\mu^{-1}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}=\bar{\Delta}+\hat{w}\tau_\mu^{-1}\mathcal{K}'\left(\frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\mathcal{K}=\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K},$$

with $s\equiv\lambda\gamma\tau_\mu^{-1}/(1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A')$. Hence, it follows from the result in Lemma A.3 that

$$h'=A\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'\left[B\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'\right]^{-1},$$

and, therefore,

$$h'\left[B\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'\right]=A\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}\right)B'.$$

Rearranging, we get

$$h'B\bar{\Delta}B'+s\hat{w}h'B\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}B'=A\bar{\Delta}B'+s\hat{w}A\mathcal{K}'\left(\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}B',$$

and right-multiplying both sides by $\left(B\bar{\Delta}B'\right)^{-1}$ yields

$$\begin{aligned} h' &= A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + s\hat{w}\left(A - h'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1} \\ &= A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + (1+\hat{w})\hat{r}\tau_\mu^{-1}A\mathcal{K}'\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1}. \end{aligned}$$

Then, from the definition of $\bar{\Delta}$ and using the fact that $\Omega_\mu = \tau_\mu\mathcal{K}\Omega\mathcal{K}'$ and $A\Gamma\Omega = \tau_\mu^{-1}A\mathcal{K}'\Omega_\mu\mathcal{K}$, it follows that

$$A\bar{\Delta} = A\left(\Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu\mathcal{K}\right) = (1+\hat{w})\,\tau_\mu^{-1}A\mathcal{K}'\Omega_\mu\mathcal{K}.$$

Plugging this back into the equation for h' we obtain the desired result,

$$h' = (1+\hat{r})\,A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1}.$$

□

Parts 1 and 2 of Proposition 3.3. Given the result in Lemma A.4, we are left with taking the limit as $T \to \infty$ of the truncated problem. In particular, we have that

$$\lim_{T\to\infty} A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} = p\left(L; w, \alpha\right), \quad \lim_{T\to\infty} A\mathcal{K}'\Omega_\eta\mathcal{K}A' = \mathbb{V}(\xi_t),$$

$$\lim_{T\to\infty}\left(A - h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}(A' - Bh') = \mathbb{V}(\xi_t - K_t), \quad \lim_{T\to\infty}\left(A - h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}A' = \mathbb{COV}\left(\xi_t - K_t, \xi_t\right),$$

$$\lim_{T\to\infty}\frac{\left(A - h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}A'}{A\mathcal{K}'\Omega_\eta\mathcal{K}A'} = 1 - \mathcal{S}.$$

Let $w \equiv \lim_{T\to\infty}\hat{w}$, and $r \equiv \lim_{T\to\infty}\hat{r}$. Then, we can show that

$$\begin{aligned} r &= \lim_{T\to\infty}\frac{\hat{w}}{1+\hat{w}}\frac{\lambda\gamma\tau_\mu A\mathcal{K}'\Omega_\eta\mathcal{K}A'}{1-\lambda\gamma\tau_\mu A\mathcal{K}'\Omega_\eta\mathcal{K}A'}\frac{\left(A - h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}A'}{A\mathcal{K}'\Omega_\eta\mathcal{K}A'} \\ &= \frac{w}{1+w}\frac{\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}{1-\lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}\left(1-\mathcal{S}\right), \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} w &= \lim_{T\to\infty}\frac{\tau_\mu}{1-\lambda\left(1-\alpha\right)\left(A-h'B\right)\mathcal{K}'\left(\Omega_\mu + \frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)\mathcal{K}\left(A'-B'h\right)} \\ &= \lim_{T\to\infty}\frac{\tau_\mu}{1-\lambda\left(1-\alpha\right)\tau_\mu\left(\left(A-h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}\left(A'-B'h\right) + \hat{r}\frac{1+\hat{w}}{\hat{w}}A\mathcal{K}'\Omega_\eta\mathcal{K}\left(A'-B'h\right)\right)} \\ &= \frac{\tau_\mu}{1-\lambda\left(1-\alpha\right)\tau_\mu\left(\mathbb{V}\left(\xi_t-K_t\right) + r\frac{1+w}{w}\left(1-\mathcal{S}\right)\mathbb{V}\left(\xi_t\right)\right)}. \end{aligned}$$

Solving for w, we obtain

$$w = \frac{\tau_\mu}{1-\lambda\left(1-\alpha\right)\tau_\mu\left(\mathbb{V}\left(\xi_t-K_t\right) + \frac{\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2}{1-\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}\right)}. \tag{A.8}$$

Lemma A.5 below establishes that $w \geq \tau_\mu$ and $r \geq 0$, which completes the proof of parts 1 and 2 of Proposition 3.3.

Lemma A.5. *If w and r satisfy equations (A.7) and (A.8), then $w \geq \tau_\mu$ and $r \geq 0$.*

Proof. The ex-ante objective of an agent i must obtain finite values under an equilibrium strategy $k_i = h' B\nu_i + h_0$. The ex-ante objective is given by

$$\begin{aligned}\mathcal{V} = & -\frac{1}{\lambda}\ln\left(\int_\mu \exp\left(-\lambda\mathbb{E}^\mu\left[u\left(-k_i, K, \theta\right)\right]\right) p\left(\mu\right) \mathrm{d}\mu\right)\\ = & \text{ constant} - \frac{1}{\lambda}\ln\left(\int_\mu \exp\left(-\frac{1}{2}\mu'\bar{S}\mu + \mu'\bar{\pi}' + \bar{\pi}\mu\right)\mathrm{d}\mu\right),\end{aligned}$$

with the matrix $\bar{S}$ and the vector $\bar{\pi}$ given by

$$\begin{aligned}\bar{S} \equiv & \ \Omega_\mu^{-1} - \lambda\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}' - \lambda\gamma\mathcal{K}A'A\mathcal{K}',\\ \bar{\pi} \equiv & -\lambda\frac{1}{2}\left(1-\alpha\right)h_0\left(A - h'B\right)\mathcal{K}' - \lambda\frac{1}{2}\chi A\mathcal{K}',\end{aligned}$$

where we used the fact that $\mathbb{E}^\mu\left[u\left(k_i, K, \theta\right)\right]$ is given by equation (A.3) and

$$p(\mu) = \left(2\pi\right)^{-u/2}\det\left(\Omega_\mu\right)^{-1/2}\exp\left(-\frac{1}{2}\mu'\Omega_\mu^{-1}\mu\right).$$

Thus, a necessary condition for $\mathcal{V}$ to be finite in equilibrium is for $\bar{S}$ to be positive definite; otherwise, the integral would become explosive.[36] Since

$$\tilde{\Omega}_\mu^{-1} = \Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}',$$

it must be that

$$\tilde{\Omega}_\mu^{-1} - \lambda\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}' \quad \text{is positive definite.}$$

Defining the vector $\digamma \equiv \left(A - h'B\right)\mathcal{K}'\tilde{\Omega}_\mu$, it follows that

$$\begin{aligned}0 \leq & \ \digamma\left(\tilde{\Omega}_\mu^{-1} - 2\lambda\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}'\right)\digamma'\\ = & \ \left(A - h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)\left(1 - \lambda\left(1-\alpha\right)\left(A - h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(A' - B'h\right)\right).\end{aligned}$$

[36] The same argument applies to how Assumption 2 ensures the problem is well defined. Specifically, a well-defined problem requires the choice set to be non-empty, which is equivalent to requiring $\bar{S}$ to be positive definite for at least one h. The necessary and sufficient condition for the existence of an h that makes $\bar{S}$ positive definite is that $\tilde{\Omega}_\mu$ is positive definite. Notice that $\tilde{\Omega}_\mu = \Omega_\mu + \frac{\lambda\gamma\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}$. It is then straightforward to see that $1 - \lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A' > 0$ is the sufficient condition to ensure that $\tilde{\Omega}_\mu$ is positive definite. Taking the limit as $T \to \infty$, this is equivalent to Assumption 2.

Let $x \equiv (A - h'B)\,\mathcal{K}'\tilde{\Omega}_\mu \mathcal{K}\,(A' - B'h)$, then we have that

$$x(1-\lambda(1-\alpha)x) \geq 0 \quad \text{or} \quad x \geq \lambda(1-\alpha)x^2 \geq 0.$$

Hence, we have that $x \geq 0$, and $1-\lambda(1-\alpha)x \geq 0$, which implies that

$$\hat{w} = \frac{\tau_\mu}{1-\lambda(1-\alpha)x} \geq \tau_\mu,$$

and, since $w = \lim_{T\to\infty} \hat{w}$, it follows that $w \geq \tau_\mu$.

Next, for a contradiction, suppose that $r < 0$. Then, it follows from equation (A.7) and Assumption 2 that $\mathbb{COV}\left(\xi_t - K_t, \xi_t\right) < 0$. Further, we have that

$$\mathbb{COV}\left(\xi_t - K_t, \xi_t\right) = \mathbb{V}\left(\xi_t\right) - (1+r)\,\mathbb{COV}\left(\hat{K}_t, \xi_t\right),$$

where $\hat{K}_t \equiv K_t/\left(1+r\right)$ is the average optimal forecast of the fundamental ξ_t under the (w,α)-modified signal process (net of the bias $\mathcal{B}$, which is uncorrelated with ξ_t),[37] so that it must be that

$$0 \leq \mathbb{COV}\left(\hat{K}_t, \xi_t\right) \leq \mathbb{V}\left(\xi_t\right).$$

Hence, $\mathbb{COV}\left(\xi_t - K_t, \xi_t\right) < 0$ implies $r > 0$ and we have a contradiction. Therefore, $r \geq 0$. □

Part 3 of Proposition 3.3. Next, we switch focus to the level of the $\mathcal{B} \equiv \lim_{T\to\infty} h_0$. From equation (A.5) and the definition of π, we have that

$$(1-\alpha)\,h_0 = \left[(1-\alpha)\,(A - \mathrm{T}B) + \alpha h'\,(B\Lambda - \mathrm{H}B)\right]\mathcal{K}'S\left[-\lambda\,(1-\alpha)\,h_0\mathcal{K}\,(A' - B'h) + \lambda\chi\mathcal{K}A'\right].$$

It is straightforward to see there exists a unique h_0 that satisfies this equation. We postulate that there exists $\tilde{\mu}$ such that

$$(1-\alpha)\,h_0 = \left[(1-\alpha)\,A + \alpha h'B\Lambda - h'B\right]\mathcal{K}'\tilde{\mu},$$

so that solving for $\tilde{\mu}$ pins down the unique h_0. To proceed, first replace the guess for h_0 on the RHS of equation (A.5),

$$\begin{aligned}
\text{RHS} \equiv\; & \left[(1-\alpha)\,(A - \mathrm{T}B) + \alpha h'\,(B\Lambda - \mathrm{H}B)\right]\mathcal{K}'S\left[-\lambda\,(1-\alpha)\,h_0\mathcal{K}\,(A' - B'h) + \lambda\chi\mathcal{K}A'\right] \\
=\; & \left[(1-\alpha)\,(A - \mathrm{T}B) + \alpha h'\,(B\Lambda - \mathrm{H}B)\right]\mathcal{K}'S \\
& \times \left\{-\lambda\mathcal{K}\,(A' - B'h)\left[(1-\alpha)\,(A - h'B) + \alpha h'B\,(\Lambda - \mathrm{I}_m)\right]\mathcal{K}'\tilde{\mu} + \lambda\chi\mathcal{K}A'\right\}
\end{aligned}$$

[37] More precisely, notice that $\hat{K}_t = p(L; w, \alpha)\int x_{it} - \mathcal{B}/(1+r)$, and that it follows from Definition 3.2 that $\int \tilde{x}_{it} = \sqrt{1+w\tau_\mu}\int x_{it}$ and $\tilde{\xi}_t = \sqrt{1+w\tau_\mu}\,\xi_t$. Therefore, $\hat{K}_t = \int \tilde{\mathbb{E}}_{it}[\xi_t] - \mathcal{B}/(1+r)$ and $\mathbb{COV}\left(\hat{K}_t, \xi_t\right) = \mathbb{COV}\left(\int \tilde{\mathbb{E}}_{it}[\xi_t], \xi_t\right)$.

Next, for the LHS of the equation, we have that

$$\text{LHS} \equiv (1-\alpha)\, h_0 = \left[(1-\alpha)\, A + \alpha h' B\Lambda - h'B\right] \mathcal{K}'\tilde{\mu},$$

and, substituting the last h using equation (A.4), it follows that

$$\begin{aligned}
\text{LHS} = &\ \left[(1-\alpha)\left(A - \text{TB}\right) + \alpha h' \left(B\Lambda - \text{H}B\right)\right] \left[\text{I}_\text{m} - \mathcal{K}'\text{S}\mathcal{K}\text{B}' \left(\text{B}\Omega\text{B}'\right)^{-1} \text{B}\right] \mathcal{K}'\tilde{\mu} \\
= &\ \left[(1-\alpha)\left(A - \text{T}B\right) + \alpha h' \left(B\Lambda - \text{H}B\right)\right] \mathcal{K}'S \left[S^{-1} - \mathcal{K}B' \left(B\Omega B'\right)^{-1} B\mathcal{K}'\right] \tilde{\mu} \\
= &\ \left[(1-\alpha)\left(A - \text{T}B\right) + \alpha h' \left(B\Lambda - \text{H}B\right)\right] \mathcal{K}'S \\
&\times \left\{\Omega_\mu^{-1} - \lambda\left[(1-\alpha)\,\mathcal{K}\left(A' - B'h\right)\left(A - h'B\right)\mathcal{K}' + \gamma\mathcal{K}A'A\mathcal{K}'\right]\right\} \tilde{\mu},
\end{aligned}$$

where the last equality uses the definition of S. Putting these results together, we have that

$$\begin{aligned}
\text{LHS} - \text{RHS} = &\ \left[(1-\alpha)\left(A - \text{TB}\right) + \alpha h' \left(B\Lambda - \text{HB}\right)\right] \mathcal{K}'S \\
&\times \left[\Omega_\mu^{-1}\tilde{\mu} + \alpha\lambda\mathcal{K}\left(A' - B'h\right) h'B\left(\Lambda - \text{I}_\text{m}\right)\mathcal{K}'\tilde{\mu} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\tilde{\mu} - \lambda\chi\mathcal{K}A'\right].
\end{aligned}$$

Since $\alpha\lambda\mathcal{K}\left(A' - B'h\right) h'B\left(\Lambda - \text{I}_\text{m}\right)\mathcal{K}' = 0$, a sufficient condition for $\text{LHS} - \text{RHS} = 0$ is

$$\Omega_\mu^{-1}\tilde{\mu} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\tilde{\mu} - \lambda\chi\mathcal{K}A' = 0,$$

which, using the Sherman-Morrison formula, implies that

$$\tilde{\mu} = \chi\lambda\left(\Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1}\mathcal{K}A' = \chi\lambda\left(\text{I}_\text{u} + \frac{\lambda\gamma\Omega_\mu\mathcal{K}\text{A}'\text{A}\mathcal{K}'}{1 - \lambda\gamma\text{A}\mathcal{K}'\Omega_\mu\mathcal{K}\text{A}'}\right)\Omega_\mu\mathcal{K}A'.$$

Therefore, we have that

$$\begin{aligned}
h_0 = &\ (1-\alpha)^{-1}\left[(1-\alpha)\, A + \alpha h'B\Lambda - h'B\right]\mathcal{K}'\tilde{\mu} \\
= &\ \left(A - h'B\right)\mathcal{K}'\tilde{\mu} \\
= &\ \left(A - h'B\right)\mathcal{K}'\chi\lambda\left(\text{I}_\text{u} + \frac{\lambda\gamma\Omega_\mu\mathcal{K}\text{A}'\text{A}\mathcal{K}'}{1 - \lambda\gamma\text{A}\mathcal{K}'\Omega_\mu\mathcal{K}\text{A}'}\right)\Omega_\mu\mathcal{K}A' \\
= &\ \chi\lambda\tau_\mu\left(A - h'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}A'\left(1 + \frac{\lambda\gamma\tau_\mu A\mathcal{K}'\Omega_\eta\mathcal{K}A'}{1 - \lambda\gamma\tau_\mu A\mathcal{K}'\Omega_\eta\mathcal{K}A'}\right).
\end{aligned}$$

Taking the limit we get

$$\mathcal{B} = \lim_{T\to\infty} h_0 = \chi\lambda\tau_\mu\mathbb{COV}(\xi_t - K_t, \xi_t)\left(1 + \frac{\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}{1 - \lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}\right) = \chi\frac{\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)}{1 - \lambda\gamma\tau_\mu\mathbb{V}\left(\xi_t\right)}\left(1 - \mathcal{S}\right),$$

which completes the proof of part 3 of the proposition. □

Proof of Proposition 3.1. Using the equivalence result from Proposition 3.3, establishing existence of an equilibrium reduces to showing that there exists a (w, r) pair that satisfies equations (A.7) and (A.8).

We start by using the intermediate value theorem to prove that there exists $w \in [\tau_\mu, \infty)$ that satisfies equation (A.8). Define

$$F(w) \equiv w \left[1 - \lambda (1-\alpha) \tau_\mu \left(\mathbb{V}(\xi_t - K_t) + \frac{\lambda \gamma \tau_\mu \mathbb{V}(\xi_t)^2 (1-\mathcal{S})^2}{1 - \lambda \gamma \tau_\mu \mathbb{V}(\xi_t)} \right) \right] - \tau_\mu,$$

such that $F(w) = 0$ implies equation (A.8). Next, notice that as $w \to \infty$, private information becomes infinitely precise and, therefore, $p(L; w, \alpha) \to a(L)$, or $K_t \to \xi_t$. It follows that $\mathcal{S} \to 1$ and $\mathbb{V}(\xi_t - K_t) \to 0$, so that $\lim_{w \to \infty} F(w) = \infty$ and there must exist some finite $\bar{w} \geq \tau_\mu$ large enough such that $F(\bar{w}) > 0$. Next, notice that when $w = \tau_\mu$,

$$F(\tau_\mu) = -\lambda (1-\alpha) \tau_\mu^2 \left(\mathbb{V}(\xi_t - K_t) + \frac{\lambda \gamma \tau_\mu \mathbb{V}(\xi_t)^2 (1-\mathcal{S})^2}{1 - \lambda \gamma \tau_\mu \mathbb{V}(\xi_t)} \right) < 0.$$

Thus, since $F(\cdot)$ is continuous, $F(\tau_\mu) < 0$, and $F(\bar{w}) > 0$, there must exist some finite $w \in [\tau_\mu, \bar{w}]$ such that $F(w) = 0$.

Further, from the definition of $\mathcal{S}$ we have that (see footnote 37)

$$1 - \mathcal{S} = \frac{\mathbb{COV}(\xi_t - K_t, \xi_t)}{\mathbb{V}(\xi_t)} \quad \Rightarrow \quad 1 - \mathcal{S} = 1 - (1+r) \frac{\mathbb{COV}(\hat{K}_t, \xi_t)}{\mathbb{V}(\xi_t)}.$$

Therefore, equation (A.7) becomes

$$r = \frac{w}{1+w} \frac{\lambda \gamma \tau_\mu \mathbb{V}(\xi_t)}{1 - \lambda \gamma \tau_\mu \mathbb{V}(\xi_t)} \left(1 - (1+r) \frac{\mathbb{COV}(\hat{K}_t, \xi_t)}{\mathbb{V}(\xi_t)} \right).$$

Since $\mathbb{COV}(\hat{K}_t, \xi_t)$ does not depend on r, the existence of w directly implies the existence of r. □

Proof of Proposition 3.5. According to equation (3.17), α affects the bias, $\mathcal{B}$, only through $1 - \mathcal{S}$. It is, then, sufficient to prove that the sensitivity, $\mathcal{S}$, is decreasing in α. Further, since $\gamma = 0$ implies $r = 0$, α affects $\mathcal{S}$ only through the endogenous scalar w. To facilitate the proof, define an alternative signal process such that

$$\xi_t = a(L)\eta_t, \qquad \text{with} \quad \eta_t \sim \mathcal{N}(0, \sigma_\eta^2), \tag{A.9}$$

$$\hat{x}_{it} = m(L)\eta_t + n(L)\hat{\epsilon}_{it}, \qquad \text{with} \quad \hat{\epsilon}_{it} \sim \mathcal{N}(0, (1-\alpha)^{-1}(1+w)^{-1}\Sigma), \tag{A.10}$$

and let the corresponding optimal Bayesian forecast be given by

$$\hat{\mathbb{E}}_{it}[\xi_t] = \hat{p}(L; w, \alpha)\hat{x}_{it}.$$

It is straightforward to show that this signal process is equivalent to the (w, α)-modified signal process for Definition 3.2, that is

$$\hat{p}(L; w, \alpha) = p(L; w, \alpha).$$

For the current proof, this signal process is more helpful. Notice that $\mathcal{S}$ is affected by α only through $p(L; w, \alpha)$, since it is defined on the basis of the objective signal process.

In what follows, we first show that

$$\lim_{\alpha \to 1^-} \frac{\mathrm{d}\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)}{\mathrm{d}\alpha} > 0.$$

We then prove, by contradiction, that there does not exist $\alpha \in [0, 1)$ such that

$$\frac{\mathrm{d}\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)}{\mathrm{d}\alpha} < 0.$$

Then, the result follows by continuity of $\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)$ with respect to α.

Step 1: $\lim_{\alpha \to 1^-} \frac{\mathrm{d}\mathbb{COV}(\xi_t - K_t, \xi_t)}{\mathrm{d}\alpha} > 0$:

It follows from equation (3.15) that $\lim_{\alpha \to 1^-} w = \tau_\mu$. So, as $\alpha \to 1^-$, the signals $\hat{x}_{it}$ become useless and, as a result,

$$\mathbb{COV}\left(K_t, \xi_t\right) = \mathbb{V}\left(K_t\right) = 0.$$

Further, since $w \geq \tau_\mu$, we have that

$$\lim_{\alpha \to 1^-} \frac{\mathrm{d}w}{\mathrm{d}\alpha} \leq 0 \quad \Rightarrow \quad \lim_{\alpha \to 1^-} \frac{\mathrm{d}\left(1 - \alpha\right)\left(1 + w\right)}{\mathrm{d}\alpha} < 0.$$

Therefore, at the limit of $\alpha \to 1^-$, an increase in α is akin to an increase in the variance of every idiosyncratic noise, which implies that (see Lemma D.2 in the Online Appendix D of Huo and Pedroni (2020)),

$$\lim_{\alpha \to 1^-} \frac{\mathrm{d}\,\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)}{\mathrm{d}\alpha} > 0.$$

Step 2: $\frac{\mathrm{d}\mathbb{COV}(\xi_t - K_t, \xi_t)}{\mathrm{d}\alpha} > 0$ for all $\alpha \in [0, 1)$:

Suppose the there exists $\alpha \in [0, 1)$ such that $\frac{\mathrm{d}\mathbb{COV}(\xi_t - K_t, \xi_t)}{\mathrm{d}\alpha} < 0$. Then, by the intermediate value theorem and continuity of $\frac{\mathrm{d}\mathbb{COV}(\xi_t - K_t, \xi_t)}{\mathrm{d}\alpha}$, there must exist some $\alpha_\dagger$ such that

$$\left.\frac{\mathrm{d}\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)}{\mathrm{d}\alpha}\right|_{\alpha = \alpha_\dagger} = 0 \quad \Rightarrow \quad \left.\frac{\mathrm{d}\left(1 - \alpha\right)\left(1 + w\right)}{\mathrm{d}\alpha}\right|_{\alpha = \alpha_\dagger} = 0 \quad \Rightarrow \quad \left.\frac{\mathrm{d}\mathbb{V}\left(\xi_t - K_t\right)}{\mathrm{d}\alpha}\right|_{\alpha = \alpha_\dagger} = 0,$$

since, for $\mathbb{COV}\left(\xi_t - K_t, \xi_t\right)$ not to change with α, it must be that the variance of the noise, $\left(1 - \alpha\right)\left(1 + w\right)$, is unchanged. Since

$$\frac{\mathrm{d}\left(1 - \alpha\right)\left(1 + w\right)}{\mathrm{d}\alpha} = -\left(1 + w\tau_\mu\right) + \left(1 - \alpha\right)\frac{\mathrm{d}w}{\mathrm{d}\alpha},$$

it follows that

$$\left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha = \alpha_\dagger} > 0.$$

However, since $\mathbb{COV}(\xi_t - K_t, \xi_t)$ and $\mathbb{V}(\xi_t - K_t)$ do not vary with α, it follows from equation (3.15) that

$$\left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger} = -\frac{\lambda\tau_\mu\left(\mathbb{V}(\xi_t - K_t) + \frac{\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2}{1-\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}\right)}{\left[1-\lambda(1-\alpha_\dagger)\tau_\mu\left(\mathbb{V}(\xi_t - K_t) + \frac{\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2}{1-\lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}\right)\right]^2} < 0.$$

Thus, we have a contradiction, and we can conclude that

$$\frac{\mathrm{d}\mathbb{COV}(\xi_t - K_t, \xi_t)}{\mathrm{d}\alpha} < 0 \;\; \Rightarrow \;\; \frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\alpha} < 0.$$

□

B Extensions

In this section, we consider three extensions to the baseline model setup. The first is the multiple-actions extension discussed in Section 3.5; here we simply provide a proof of the results presented there. The second extension allows for a more general utility specification, which covers economies with different forms of inefficiencies. The third extension is to the information structure, allowing the fundamental to depend on multiple aggregate shocks.

B.1 Multiple actions

In this section, we extend the baseline setup to allow for multiple actions instead of just a single one. Each agent i takes J actions, so that $k_{it} \in \mathrm{R}^J$. In what follows, we first demonstrate that the utility specification with multiple actions introduced in Section 3.5, equation (3.18), represents an efficient economy under both complete and incomplete information, provided there is no concern for ambiguity. We then proceed to present the proof of Proposition 3.6, which characterizes the equilibrium when there is ambiguity and ambiguity aversion under this multiple actions setup.

B.1.1 An Efficient Economy

Consider the following extension to multiple actions of the generic quadratic utility specification from Angeletos and Pavan (2007):

$$u\left(k_{it}, K_t, \Sigma_t, \xi_t\right) = \frac{1}{2}k_{it}'U_{kk}k_{it} + \frac{1}{2}K_t'U_{KK}K_t + \frac{1}{2}\xi_t'U_{\xi\xi}\xi_t + \frac{1}{2}\Sigma_t'U_{\Sigma\Sigma}\Sigma_t + \xi_t'U_{k\xi}'k_{it} + K_t'U_{kK}'k_{it} + \xi_t'U_{K\xi}'K_t$$
$$+ U_k k_{it} + U_K K_t + U_\xi \xi_t + \text{const.},$$

where K_t and Σ_t denote respectively the cross-sectional mean and dispersion of the J actions,

$$K_t \equiv \int_i k_{it}\,\mathrm{d}i, \quad \text{and} \quad \Sigma_t \equiv \left(\sqrt{\int_i \left(k_{1,it} - K_{1,t}\right)^2 \mathrm{d}i}, \cdots, \sqrt{\int_i \left(k_{j,it} - K_{j,t}\right)^2 \mathrm{d}i}, \cdots, \sqrt{\int_i \left(k_{J,it} - K_{J,t}\right)^2 \mathrm{d}i}\right).$$

The jth elements of k_{it} and K_t are represented by $k_{j,it}$ and $K_{j,t}$, respectively. We assume that $U_{\Sigma\Sigma}$ is diagonal, and that the information structure is the same as in the single-action setup.

Equilibrium. Without any concern for ambiguity, we now define and characterize an equilibrium for this model.

Definition B.1. *In the absence of ambiguity, an equilibrium is a strategy $k\left(x_i^t\right)$ such that*

$$k\left(x_i^t\right) = \operatorname{argmax} k\, \mathbb{E}\left[u\left(k, K\left(\eta^t\right), \Sigma\left(\eta^t\right), \xi\left(\eta^t\right)\right) \mid x_i^t\right],$$

where $K\left(\eta^t\right) \equiv \int_i k\left(x_i^t\right)\mathrm{d}i$ denotes the equilibrium aggregate action, and

$$\Sigma\left(\eta^t\right) \equiv \left(\sigma_1\left(\eta^t\right), \cdots, \sigma_j\left(\eta^t\right), \cdots, \sigma_J\left(\eta^t\right)\right)', \quad \textit{with} \quad \sigma_j\left(\eta^t\right) \equiv \sqrt{\int_i \left(k_{j,i}\left(x_i^t\right) - K_j\left(\eta^t\right)\right)^2 \mathrm{d}i},$$

denotes the equilibrium cross-sectional dispersion of actions.

Proposition B.1. *In the absence of ambiguity, a strategy $k\left(x_i^t\right)$ is an equilibrium under incomplete information if and only if*

$$k\left(x_i^t\right) = \left(\mathrm{I}-\Theta\right)\mathbb{E}\left[\kappa\left(\xi_t\right) \mid x_i^t\right] + \Theta\,\mathbb{E}\left[K\left(\eta^t\right) \mid x_i^t\right],$$

where the equilibrium degree of coordination is captured by the $J \times J$ matrix

$$\Theta \equiv -U_{kk}^{-1}U_{kK},$$

and $\kappa\left(\xi_t\right)$ denotes the equilibrium allocation under complete information, given by

$$\kappa\left(\xi_t\right) \equiv \underbrace{-\left(U_{kk}+U_{kK}\right)^{-1}U_{k\xi}}_{\kappa}\ \xi_t\ \underbrace{-\left(U_{kk}+U_{kK}\right)^{-1}U_k'}_{\kappa_0}.$$

Proof. We first characterize the complete-information benchmark. Let $\mathcal{I}_{it}$ be the information set of agent i in period t. Under complete information, we have that $\xi_t \in \mathcal{I}_{it}$. That is, all agents have perfect information about both the fundamental ξ_t and, consequently, about the aggregate action K_t. The agent's first-order condition is then given by

$$\frac{\partial u\left(k_{it}, K_t, \Sigma_t, \xi_t\right)}{\partial k_{it}} = k_{it}'U_{kk} + \xi_t'U_{k\xi}' + K_t'U_{kK}' + U_k = 0.$$

Using the fact that $k_{it} = K_t$, the equilibrium strategy under complete information is such that

$$k_{it} = \kappa\left(\xi_t\right) \equiv \underbrace{-\left(U_{kk}+U_{kK}\right)^{-1}U_{k\xi}}_{\kappa}\ \xi_t\ \underbrace{-\left(U_{kk}+U_{kK}\right)^{-1}U_k'}_{\kappa_0},$$

where both κ and κ_0 are $J \times 1$ vectors.

When information is incomplete, the agent's first-order condition becomes

$$-U_{kk}\ k_{it} = U_{k\xi}\ \mathbb{E}\left[\xi_t \mid x_i^t\right] + U_{kK}\ \mathbb{E}\left[K_t \mid x_i^t\right] + U_k'.$$

Multiplying $-\left(U_{kk}+U_{kK}\right)^{-1}$ to both sides of the equation implies

$$\left(U_{kk}+U_{kK}\right)^{-1}U_{kk}\ k_{it} = -\left(U_{kk}+U_{kK}\right)^{-1}U_{k\xi}\ \mathbb{E}\left[\xi_t \mid x_i^t\right] - \left(U_{kk}+U_{kK}\right)^{-1}U_{kK}\ \mathbb{E}\left[K_t \mid x_i^t\right] - \left(U_{kk}+U_{kK}\right)^{-1}U_k',$$

and it follows that

$$k_{it} = U_{kk}^{-1}\left(U_{kk}+U_{kK}\right)\ \mathbb{E}\left[\kappa\left(\xi_t\right) \mid x_i^t\right] - U_{kk}^{-1}U_{kK}\ \mathbb{E}\left[K_t \mid x_i^t\right].$$

which completes the proof. □

Efficient allocation. Abstracting from ambiguity concerns, an efficient allocation is the strategy $k\left(x_i^t\right)$ that maximizes ex-ante utility, subject only to the constraint that the private information of any agent cannot be transferred to any other agent.

Definition B.2. *In the absence of ambiguity, an efficient allocation is a strategy $k\left(x_i^t\right)$ that maximizes ex-ante expected utility,*

$$\mathbb{E}\left[u\left(k, K\left(\eta^t\right), \Sigma\left(\eta^t\right), \xi\left(\eta^t\right)\right)\right].$$

Proposition B.2. *In the absence of ambiguity, a strategy $k\left(x_i^t\right)$ is efficient under incomplete information if and only if*

$$k\left(x_i^t\right) = (\mathrm{I} - \Theta^*) \int_{\eta^t} \kappa^*(\xi\left(\eta^t\right)) \mathrm{d}P\left(\eta^t \mid x_i^t\right) + \Theta^* \int_{\eta^t} K\left(\eta^t\right) \mathrm{d}P\left(\eta^t \mid x_i^t\right),$$

where $P\left(\eta^t \mid x_i^t\right)$ denotes the cumulative distribution function of η_t conditional on x_i^t, the efficient degree of coordination is captured by the $J \times J$ matrix

$$\Theta^* = -\left(U_{kk} + U_{\Sigma\Sigma}\right)^{-1}\left(U_{KK} + U_{kK} + U_{kK}' - U_{\Sigma\Sigma}\right),$$

and $\kappa^*(\xi_t)$ denotes the efficient allocation under complete information, given by

$$\kappa^*(\xi_t) \equiv \underbrace{-\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right)^{-1}\left(U_{k\xi} + U_{K\xi}\right)}_{\kappa^*} \xi_t \underbrace{-\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right)^{-1}\left(U_k + U_K\right)'}_{\kappa_0^*}.$$

Proof. We first characterize the first-best allocation, that is, the efficient allocation under complete information. Let $\mathcal{I}_{it}$ be the information set of agent i in period t. Under complete information, we have that $\xi_t \in \mathcal{I}_{it}$. It is, then, straightforward to show that the first-best allocation features $k_{it} = K_t$, which implies that $\Sigma_t = 0$. It follows that the efficient level of K_t must maximize

$$\frac{1}{2} K_t'\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right) K_t + \frac{1}{2}\xi_t' U_{\xi\xi}\xi_t + \xi_t'\left(U_{k\xi} + U_{K\xi}\right)' K_t + \left(U_k + U_K\right) K_t + U_\theta \xi_t + \text{const.},$$

which implies the following first-order condition,

$$K_t'\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right) + \xi_t'\left(U_{k\xi} + U_{K\xi}\right)' + \left(U_k + U_K\right) = 0.$$

It follows that the efficient allocation satisfies

$$k_{it} = K_t = \underbrace{-\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right)^{-1}\left(U_{k\xi} + U_{K\xi}\right)}_{\kappa^*} \xi_t \underbrace{-\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right)^{-1}\left(U_k + U_K\right)'}_{\kappa_0^*},$$

where both κ^* and κ_0^* are $J \times 1$ vectors.

To characterize the efficient allocation under incomplete information, define the Lagrangian of the problem in

Definition B.2 by

$$\Lambda = \int_{\eta^t}\int_{x_i^t} u\left(k\left(x_i^t\right), K\left(\eta^t\right), \Sigma\left(\eta^t\right), \xi\left(\eta^t\right)\right) \mathrm{d}P\left(x_i^t \mid \eta^t\right) \mathrm{d}P\left(\eta^t\right)$$
$$+ \int_{\eta^t} \iota\left(\eta^t\right)\left[K\left(\eta^t\right) - \int_{x_i^t} k\left(x_i^t\right) \mathrm{d}P\left(x_i^t \mid \eta^t\right)\right] \mathrm{d}P\left(\eta^t\right)$$
$$+ \int_{\eta^t} \sum_{j=1}^{J} \varphi_j\left(\eta^t\right)\left[\sigma_j^2\left(\eta^t\right) - \int_{x_i^t}\left(k_{j,i}\left(x_i^t\right) - K_j\left(\eta^t\right)\right)^2 P\left(x_i^t \mid \eta^t\right) \mathrm{d}x_i^t\right] \mathrm{d}P\left(\eta^t\right),$$

where $\iota\left(\eta^t\right)$ and $\varphi_j\left(\eta^t\right)$ denote the multipliers on the definitions of $K\left(\eta^t\right)$ and $\sigma_j\left(\eta^t\right)$, respectively. Further, $P\left(x_i^t \mid \eta^t\right)$ denotes the CDF of x_i^t conditional on η_t, and and $P\left(\eta^t\right)$ denotes the unconditional CDF of η^t.

To ease notation, denote $\varphi\left(\eta^t\right) \equiv \operatorname{diag}\left(\varphi_1\left(\eta^t\right), \cdots, \varphi_j\left(\eta^t\right), \cdots, \varphi_J\left(\eta^t\right)\right)$. Then, the first-order conditions can be written as

$$\int_{x_i^t}\left(\frac{\partial u(\cdot)}{\partial K} + \iota\left(\eta^t\right) + 2\varphi\left(\eta^t\right)\left(k\left(x_i^t\right) - K\left(\eta^t\right)\right)\right) \mathrm{d}P\left(x_i^t \mid \eta^t\right) = 0, \quad \text{for almost all } \eta^t, \tag{B.1}$$

$$\int_{\eta^t}\left(\frac{\partial u(\cdot)}{\partial k} - \iota\left(\eta^t\right) - 2\varphi\left(\eta^t\right)\left(k\left(x_i^t\right) - K\left(\eta^t\right)\right)\right) \mathrm{d}P\left(\eta^t \mid x_i^t\right) = 0, \quad \text{for almost all } x_i^t, \tag{B.2}$$

$$\int_{x_i^t}\left(\frac{\partial u(\cdot)}{\partial \Sigma}\right) \mathrm{d}P\left(x_i^t \mid \eta^t\right) + 2\varphi\left(\eta^t\right)\Sigma\left(\eta^t\right) = 0, \quad \text{for almost all } \eta^t. \tag{B.3}$$

Rearranging equations (B.1) and (B.3), we obtain

$$\int_{x_i^t} \frac{\partial u(\cdot)}{\partial K} \mathrm{d}P\left(x_i^t \mid \eta^t\right) + \iota\left(\eta^t\right) = 0, \quad \text{and} \quad \varphi\left(\eta^t\right) = -\frac{1}{2} U_{\Sigma\Sigma}, \quad \text{for almost all } \eta^t.$$

Further, since

$$\frac{\partial u(\cdot)}{\partial K} = U_{KK} K\left(\eta^t\right) + U_{kK}' \, k\left(x_i^t\right) + U_{K\xi}\, \xi\left(\eta^t\right) + U_K,$$

it follows that

$$\iota\left(\eta^t\right) = -\left(U_{KK} + U_{kK}'\right) K\left(\eta^t\right) - U_{K\xi}\, \xi\left(\eta^t\right) - U_K.$$

Using these two expressions to replace $\iota\left(\eta^t\right)$ and $\varphi\left(\eta^t\right)$ in equation (B.2), and using the fact that

$$\frac{\partial u(\cdot)}{\partial k} = U_{kk} k\left(x_i^t\right) + U_{kK} K\left(\eta^t\right) + U_{k\xi}\xi\left(\eta^t\right) + U_k,$$

yields

$$k\left(x_i^t\right) = \left(U_{kk} + U_{\Sigma\Sigma}\right)^{-1}\left(U_{kk} + U_{kK} + U_{kK}' + U_{KK}\right) \int_{\eta^t} \kappa^*(\xi\left(\eta^t\right)) \mathrm{d}P\left(\eta^t \mid x_i^t\right)$$
$$- \left(U_{kk} + U_{\Sigma\Sigma}\right)^{-1}\left(U_{KK} + U_{kK} + U_{kK}' - U_{\Sigma\Sigma}\right) \int_{\eta^t} K\left(\eta^t\right) \mathrm{d}P\left(\eta^t \mid x_i^t\right),$$

which completes the proof. □

By comparing Propositions B.1 and B.2, we arrive at the following corollary.

Corollary B.1. *An economy is efficient if and only if*

$$\kappa\left(\xi_t\right) = \kappa^*(\xi_t), \quad \textit{and} \quad \Theta = \Theta^*.$$

Next, notice that the utility specification in equation (3.18), used in Section 3.5,

$$u(k_{it}, K_t, \xi_t) = \frac{1}{2}\left(k_{it} - \kappa\xi_t\right)' \Psi_k \left(k_{it} - \kappa\xi_t\right) + \frac{1}{2}\left(k_{it} - K_t\right)' \Psi_K \left(k_{it} - K_t\right) + \chi\xi_t - \frac{1}{2}\gamma\xi_t^2,$$

implies that

$$U_K = 0, \quad U_{\Sigma\Sigma} = U_{K\xi} = 0, \quad \text{and} \quad U_{kK} = U_{kK'} = U_{KK}.$$

These constraints imply the conditions from Corollary B.1, which then leads to following result.

Claim 1. *The economy with utility given by equation* (3.18) *is efficient under both complete and incomplete information.*

We conclude this subsection by two additional remarks:

1. We can normalize $U_k = 0$, and thus, $\kappa_0 = 0$ without loss of generality. A nonzero U_k would only add an exogenous vector of constants to the action strategy under complete or incomplete information. This same exogenous vector of constants also applies to the equilibrium action strategy with ambiguity. This vector of constants can be regarded as the deterministic steady state of the economy, which can always be abstracted away by redefining actions as deviations from the deterministic steady state.
2. We demonstrate that economy with the utility specified as in equation (3.18) is efficient. This statement can be strengthened in the sense that, as long as $U_{\Sigma\Sigma} = 0$, equation (3.18) is the only utility specification that ensures efficiency under complete and incomplete information.

B.1.2 Equilibrium with Ambiguity

We now proceed to characterize the equilibrium with ambiguity. First notice that the utility specified in equation (3.18) is equivalent to the generic quadratic utility if we set

$$U_{kk} = \Psi_k + \Psi_K, \quad \text{and} \quad U_{KK} = \Psi_K.$$

From this point forward, we use these conditions to switch to the notation used in the paper, with Ψ_k and Ψ_K.

Analogously to Proposition 3.2, it can be shown that the optimal strategies for the vector of J actions of all agents are such that

$$k_{it} = \left(\mathrm{I} - \Theta\right) \mathcal{F}_{it}\left[\kappa\xi_t\right] + \Theta\, \mathcal{F}_{it}\left[K_t\right], \tag{B.4}$$

where $\mathcal{F}_{it}[\cdot]$ represents agent i's subjective expectation operator, that is,

$$\mathcal{F}_{it}[\cdot] \equiv \int_{\mu^t} \mathbb{E}^{\mu^t}[\,\cdot\, \mid x_i^t]\, \hat{p}\left(\mu^t \mid x_i^t\right) \mathrm{d}\mu^t, \quad \text{with} \quad \hat{p}\left(\mu^t \mid x_i^t\right) \propto \phi'\left(\mathbb{E}^{\mu^t}[u\left(k_{it}, K_t, \xi_t\right)]\right) p\left(\mu^t \mid x_i^t\right).$$

Moreover, the coordination matrix, Θ, is such that

$$\Theta = U_{kk}^{-1} U_{KK} = \left(\Psi_k + \Psi_K\right)^{-1} \Psi_K.$$

B.1.3 Proof of Proposition 3.6

Next, as in the single action case, we consider a truncated version of the problem using exactly the same notation as in the proof of Proposition 3.3. We identify a specific form for the equilibrium optimal strategies, which we then use to prove the main equivalence result, Proposition 3.6.

Define higher-order subjective expectations recursively as follows:

$$\bar{\mathcal{F}}^n[\mathrm{X}] \equiv \begin{cases} \mathrm{X}, & \text{if} \quad n = 0; \\ \int_i \mathcal{F}_i\left[\bar{\mathcal{F}}^{n-1}[\mathrm{X}]\right] \mathrm{d}i, & \text{if} \quad n \geq 1. \end{cases}$$

By iteratively eliminating $\mathcal{F}_{it}[K]$ in the best response (B.4), we obtain

$$k_i = \sum_{m=0}^{\infty} \Theta^m \left(I - \Theta\right) \kappa\, \mathcal{F}_i\left[\bar{\mathcal{F}}^m[\theta]\right].$$

Notice that, as long as subjective expectations are Gaussian, agent i's subjective expectations about any order must be linear in signals, that is,

$$\mathcal{F}_i\left[\bar{\mathcal{F}}^m[\theta]\right] = \tilde{h}_m'\, x_i + \tilde{q}_m,$$

where the $J \times n$ matrix $\tilde{h}_m'$ and $J \times 1$ vector $\tilde{q}_m$ represent the sensitivity and bias of the mth-order subjective expectation. Further, let the eigenvalue decomposition of Θ be given by

$$\Theta \equiv \sum_{j=1}^{J} \alpha_j Q^{-1} e_j e_j' Q,$$

where e_j denotes the j-th column of a $J \times J$ identity matrix. It follows that

$$\begin{aligned} k_i &= \sum_{m=0}^{\infty} \left(\sum_{j=1}^{J} \alpha_j Q^{-1} e_j e_j' \, Q \right)^m \left(\sum_{j=1}^{J} (1-\alpha_j) \, Q^{-1} e_j e_j' \, Q \right) \kappa \left(\tilde{h}_m' \, x_i + \tilde{q}_m \right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{j=1}^{J} \alpha_j^m Q^{-1} e_j e_j' \, Q \right) \left(\sum_{j=1}^{J} (1-\alpha_j) \, Q^{-1} e_j e_j' \, Q \right) \kappa \left(\tilde{h}_m' \, x_i + \tilde{q}_m \right) \\ &= \sum_{m=0}^{\infty} \sum_{j=1}^{J} (1-\alpha_j) \, \alpha_j^m Q^{-1} e_j e_j' \, Q \, \kappa \left(\tilde{h}_m' \, x_i + \tilde{q}_m \right) \\ &= \sum_{j=1}^{J} Q^{-1} e_j e_j' \, Q \, \kappa \left((1-\alpha_j) \sum_{m=0}^{\infty} \alpha_j^m \tilde{h}_m' \, x_i + (1-\alpha_j) \sum_{m=0}^{\infty} \alpha_j^m \tilde{q}_m \right) \\ &= \sum_{j=1}^{J} Q^{-1} e_j e_j' \, Q \, \kappa \left(\hat{h}_j' \, x_i + \hat{q}_j \right), \end{aligned}$$

where $\hat{h}_j$ and $\hat{q}_j$ are defined as

$$\hat{h}_j \equiv (1-\alpha_j) \sum_{m=0}^{\infty} \alpha_j^m \tilde{h}_m', \quad \text{and} \quad \hat{q}_j \equiv (1-\alpha_j) \sum_{m=0}^{\infty} \alpha_j^m \tilde{q}_m.$$

Interpret $\kappa \left(\hat{h}_j' \, x_i + \hat{q}_j \right)$, for all j, as a set of forecasting rules for the equilibrium allocation under complete information, $\kappa\theta$. Then, the derived expression implies that the optimal strategy for each of the J actions is a linear combination of these forecasting rules. This linear relationship can be "orthogonalized" by transforming the actions k_i and the complete information allocation $\kappa\theta$ using the matrix Q. Specifically, let

$$\hat{k}_i \equiv Q \, k_i, \quad \text{and} \quad \hat{\kappa} \equiv Q \, \kappa.$$

It follows that

$$\hat{k}_i = \sum_{j=1}^{J} e_j e_j' \, \hat{\kappa} \left(\hat{h}_j' x_i + \hat{q}_j \right),$$

so that the j-th transformed action, the j-th row of $\hat{k}_i$, is equal to $e_j' \, \hat{\kappa} \left(\hat{h}_j' \, x_i + \hat{q}_j \right)$.

By defining

$$\mathcal{H} \equiv \left[\begin{array}{cccc} e_1' \hat{\kappa} \hat{h}_1' & e_2' \hat{\kappa} \hat{h}_2' & \ldots & e_J' \hat{\kappa} \hat{h}_J' \end{array} \right], \quad \text{and} \quad \mathcal{Q} \equiv \left[\begin{array}{cccc} e_1' \hat{\kappa} \hat{q}_1 & e_2' \hat{\kappa} \hat{q}_2 & \ldots & e_J' \hat{\kappa} \hat{q}_J \end{array} \right]',$$

the expression for $\hat{k}_i$ can be compactly written as

$$\hat{k}_i = \mathcal{H}' B \nu_i + \mathcal{Q}.$$

Similarly, the Q-transformed version of the complete information solution can be written as

$$\hat{\kappa}(\theta) = \hat{\kappa} A \nu_i = \sum_{j=1}^{J} e_j e_j' \hat{\kappa} A \nu_i = \mathcal{A} \nu_i, \quad \text{with} \quad \mathcal{A} \equiv \begin{bmatrix} e_1' \hat{\kappa} A & e_2' \hat{\kappa} A & \dots & e_J' \hat{\kappa} A \end{bmatrix}' = \hat{\kappa} \otimes A.$$

Further, the utility function can also be transformed in a similar way,

$$u_i = \frac{1}{2} \left(\hat{k}_i - \hat{\kappa}(\theta) \right)' \hat{\Psi}_k \left(\hat{k}_i - \hat{\kappa}(\theta) \right) + \frac{1}{2} \left(\hat{k}_i - \hat{K} \right)' \hat{\Psi}_K \left(\hat{k}_i - \hat{K} \right) + \chi\theta - \frac{1}{2}\gamma\theta^2 + \text{const.},$$

with

$$\hat{\Psi}_k \equiv \left(Q^{-1} \right)' \Psi_k Q^{-1}, \quad \text{and} \quad \hat{\Psi}_K \equiv \left(Q^{-1} \right)' \Psi_K Q^{-1}.$$

It follows that

$$\begin{aligned} \mathbb{E}^{\mu}[u_i] = & \frac{1}{2}\mu' \mathcal{K} \left(\mathcal{H}' B - \mathcal{A} \right)' \hat{\Psi}_k \left(\mathcal{H}' B - \mathcal{A} \right) \mathcal{K}' \mu - \frac{1}{2}\gamma \mu' \mathcal{K} A' A \mathcal{K}' \mu + \\ & \frac{1}{2}\mu' \mathcal{K} \left(\mathcal{H}' B - \mathcal{A} \right)' \hat{\Psi}_K \mathcal{Q} + \frac{1}{2}\mathcal{Q}' \hat{\Psi}_k \left(\mathcal{H}' B - \mathcal{A} \right) \mathcal{K}' \mu + \frac{1}{2}\chi A \mathcal{K}' \mu + \frac{1}{2}\chi \mu' \mathcal{K} A'. \end{aligned}$$

Thus, the distorted subjective belief must satisfy

$$\hat{p}(\mu \mid x_i) \propto \exp\left(-\frac{1}{2}\mu' S^{-1} \mu + \frac{1}{2}\mu' S^{-1} \left(M x_i + \Pi \right) + \frac{1}{2} \left(M x_i + \Pi \right)' S^{-1} \mu \right),$$

with matrices S, M, and Π given by

$$\begin{aligned} S &\equiv \left(\mathcal{K} B' \left(B \Omega B' \right)^{-1} B \mathcal{K}' + \Omega_{\mu}^{-1} + \lambda \left(\mathcal{K} \left(\mathcal{H}' B - \mathcal{A} \right)' \hat{\Psi}_k \left(\mathcal{H}' B - \mathcal{A} \right) \mathcal{K}' - \gamma \mathcal{K} A' A \mathcal{K}' \right) \right)^{-1}. \\ M &\equiv S \mathcal{K} \left(B \Omega B' \right)^{-1}, \quad \text{and} \quad \Pi \equiv S \left(-\lambda \mathcal{K} \left(\mathcal{H}' B - \mathcal{A} \right)' \hat{\Psi}_k \mathcal{Q} - \lambda \chi \mathcal{K} A' \right). \end{aligned}$$

From agent i's first order condition, equation (B.4), we have that

$$\hat{k}_i = \left(\mathrm{I} - \sum_{j=1}^{J} \alpha_j e_j e_j' \right) \mathcal{A} \mathcal{F}_i[\nu_i] + \left(\sum_{j=1}^{J} \alpha_j e_j e_j' \right) \mathcal{F}_i\left[\hat{K} \right],$$

and, therefore,

$$\mathcal{H}' B \nu_i + \mathcal{Q} = \left(\mathrm{I} - \sum_{j=1}^{J} \alpha_j e_j e_j' \right) \mathcal{A} \mathcal{F}_i[\nu_i] + \left(\sum_{j=1}^{J} \alpha_j e_j e_j' \right) \left(\mathcal{H}' B \Lambda \mathcal{F}_i[\nu_i] + \mathcal{Q} \right).$$

Moreover, the distorted subjective expectations satisfy

$$\begin{aligned}
\mathcal{F}_i[\nu_i] &= \int_\mu \mathbb{E}^\mu[\nu_i|x_i]\,\hat{p}(\mu|x_i)\,\mathrm{d}\mu \\
&= \int_\mu \left(\mathbb{E}^\mu[\nu_i-\mu|x_i]+\mu\right)\hat{p}(\mu|x_i)\,\mathrm{d}\mu \\
&= \int_\mu \left(\Omega B'(B\Omega B')^{-1}(x_i-\mu)+\mu\right)\hat{p}(\mu|x_i)\,\mathrm{d}\mu \\
&= \Omega B'(B\Omega B')^{-1}x_i+\left(\mathrm{I}-\Omega B'(B\Omega B')^{-1}B\right)\mathcal{K}'\int_\mu \mu\hat{p}_i(\mu)\,\mathrm{d}\mu \\
&= \Omega B'(B\Omega B')^{-1}x_i+\left(\mathrm{I}-\Omega B'(B\Omega B')^{-1}B\right)\mathcal{K}'S\mathcal{K}B'(B\Omega B')^{-1}x_i \\
&\quad+\left(\mathrm{I}-\Omega B'(B\Omega B')^{-1}B\right)\mathcal{K}'S\left(-\lambda\mathcal{K}(\mathcal{H}'B-\mathcal{A})'\hat{\Psi}_k\mathcal{Q}-\lambda\chi\mathcal{K}A'\right).
\end{aligned}$$

Matching coefficients then implies that

$$\mathcal{H}'=(\mathrm{I}-\Phi)\,\mathrm{T}+\Phi\mathcal{H}'\mathrm{H}+\left[(\mathrm{I}-\Phi)(\mathcal{A}-\mathrm{T}B)+\Phi\mathcal{H}'(B\Lambda-\mathrm{H}B)\right]\mathcal{K}'S\mathcal{K}B'(B\Omega B')^{-1}, \tag{B.5}$$

and

$$(\mathrm{I}-\Phi)\,\mathcal{Q}=\left[(\mathrm{I}-\Phi)(\mathcal{A}-\mathrm{T}B)+\Phi\mathcal{H}'(B\Lambda-\mathrm{H}B)\right]\mathcal{K}'S\left(-\lambda\mathcal{K}(\mathcal{H}'B-\mathcal{A})'\hat{\Psi}_k\mathcal{Q}-\lambda\chi\mathcal{K}A'\right), \tag{B.6}$$

where T, H, and Φ are given by

$$\mathrm{T}\equiv\mathcal{A}\Omega B'(B\Omega B')^{-1},\quad \mathrm{H}\equiv B\Lambda\Omega B'(B\Omega B')^{-1},\quad \text{and}\quad \Phi\equiv\sum_{j=1}^{J}\alpha_j e_j e_j'.$$

In what follows, we first focus on equation (B.5). Through a sequence of lemmas, we show that this fixed-point problem for $\mathcal{H}$ can be recast as the linear combination of pure forecasting problems. We then proceed to characterize $\mathcal{Q}$ using equation (B.6).

Lemma B.1. *Define*

$$\hat{\Omega}\equiv\Omega+\mathcal{K}'\mathcal{W}\mathcal{K},\quad \hat{\mathrm{T}}\equiv\mathcal{A}\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1},\quad \hat{\mathrm{H}}\equiv B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1},$$

and

$$\mathcal{W}\equiv\left(\Omega_\mu^{-1}+\lambda\left(\mathcal{K}(\mathcal{H}'B-\mathcal{A})'\hat{\Psi}_k(\mathcal{H}'B-\mathcal{A})\mathcal{K}'-\gamma\mathcal{K}A'A\mathcal{K}'\right)\right)^{-1}.$$

Then, the equilibrium $\mathcal{H}$ solves the following fixed-point problem

$$\mathcal{H}'=(\mathrm{I}-\Phi)\,\hat{\mathrm{T}}+\Phi\mathcal{H}'\hat{\mathrm{H}}.$$

Proof. Using the Woodbury matrix identity, we have that

$$
\begin{aligned}
\left(B\hat{\Omega}B'\right)^{-1} &= \left(B\Omega B' + B\mathcal{K}'\mathcal{W}\mathcal{K}B'\right)^{-1} \\
&= \left(B\Omega B'\right)^{-1} - \left(B\Omega B'\right)^{-1} B\mathcal{K}' \left(\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}' + \mathcal{W}^{-1}\right)^{-1} \mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
&= \left(B\Omega B'\right)^{-1} - \left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1}.
\end{aligned} \tag{B.7}
$$

If some $\tilde{\mathcal{H}}$ is such that $\tilde{\mathcal{H}}' = (\mathrm{I}-\Phi)\,\hat{\mathrm{T}} + \Phi\tilde{\mathcal{H}}'\hat{\mathrm{H}}$, then

$$
\begin{aligned}
\tilde{\mathcal{H}}' =\ & (\mathrm{I}-\Phi)\,\mathcal{A}\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} + \Phi\tilde{\mathcal{H}}'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} \\
=\ & (\mathrm{I}-\Phi)\,\mathcal{A}\left(\Omega + \mathcal{K}'\mathcal{W}\mathcal{K}\right)B'\left(B\hat{\Omega}B'\right)^{-1} + \Phi\tilde{\mathcal{H}}'B\Lambda\left(\Omega + \mathcal{K}\mathcal{W}\mathcal{K}'\right)B'\left(B\hat{\Omega}B'\right)^{-1} \\
=\ & (\mathrm{I}-\Phi)\,\mathcal{A}\Omega B'\left(B\hat{\Omega}B'\right)^{-1} + (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\hat{\Omega}B'\right)^{-1} \\
& + \Phi\tilde{\mathcal{H}}'B\Lambda\Omega B'\left(B\hat{\Omega}B'\right)^{-1} + \Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\hat{\Omega}B'\right)^{-1}.
\end{aligned}
$$

Using equation (B.7), it follows that

$$
\begin{aligned}
\tilde{\mathcal{H}}' =\ & (\mathrm{I}-\Phi)\,\mathcal{A}\Omega B'\left(B\Omega B'\right)^{-1} - (\mathrm{I}-\Phi)\,\mathcal{A}\Omega B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
& + (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
& + \Phi\tilde{\mathcal{H}}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1} - \Phi\tilde{\mathcal{H}}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
& + \Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} - \Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
=\ & \underbrace{(\mathrm{I}-\Phi)\,\mathcal{A}\Omega B'\left(B\Omega B'\right)^{-1}}_{(\mathrm{I}-\Phi)\mathrm{T}} + \underbrace{\Phi\tilde{\mathcal{H}}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1}}_{\Phi\tilde{\mathcal{H}}'\mathrm{H}} \\
& - \underbrace{(\mathrm{I}-\Phi)\,\mathcal{A}\Omega B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1}}_{(\mathrm{I}-\Phi)\mathrm{T}B\mathcal{K}'\mathcal{S}\mathcal{K}B'(B\Omega B')^{-1}} - \underbrace{\Phi\tilde{\mathcal{H}}'B\Lambda\Omega B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1}}_{\Phi\tilde{\mathcal{H}}'\mathrm{H}B\mathcal{K}'\mathcal{S}\mathcal{K}B'(B\Omega B')^{-1}} \\
& + (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
& + \Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} - \Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1}.
\end{aligned}
$$

Further, notice that the terms in the second-to-last line can be rewritten as

$$
\begin{aligned}
& (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} - (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
=\ & (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}' + \mathcal{W}^{-1}\right)\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}' + \mathcal{W}^{-1}\right)^{-1} \mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
& - (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}'\left(\mathcal{K}B'\left(B\Omega B'\right)^{-1} B\mathcal{K}' + \mathcal{W}^{-1}\right)^{-1} \mathcal{K}B'\left(B\Omega B'\right)^{-1} \\
=\ & (\mathrm{I}-\Phi)\,\mathcal{A}\mathcal{K}'\mathcal{S}\mathcal{K}B'\left(B\Omega B'\right)^{-1},
\end{aligned}
$$

and, similarly, the terms in the last line can be rewritten as

$$\begin{aligned}&\Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1}-\Phi\hat{\mathcal{H}}'B\Lambda\mathcal{K}'\mathcal{W}\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}\\&=\Phi\tilde{\mathcal{H}}'B\Lambda\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1}.\end{aligned}$$

Therefore, we have that

$$\tilde{\mathcal{H}}'=\left(\mathrm{I}-\Phi\right)\mathrm{T}+\Phi\tilde{\mathcal{H}}'\mathrm{H}+\Big[\left(\mathrm{I}-\Phi\right)\left(\mathcal{A}-\mathrm{T}B\right)+\Phi\tilde{\mathcal{H}}'\left(B\Lambda-\mathrm{H}B\right)\Big]\mathcal{K}'S\mathcal{K}B'\left(B\Omega B'\right)^{-1},$$

which is equivalent to the expression for $\mathcal{H}$ in equation (B.5). □

Lemma B.2. *For any $j\in\{1,\cdots,J\}$, define*

$$\Omega_{\Gamma_j}\equiv\Gamma_j\,\hat{\Omega},\quad \textit{with}\quad \Gamma_j\equiv\begin{bmatrix}\mathrm{I}_u & 0_{u,m-u}\\ 0_{m-u,u} & \frac{\mathrm{I}_{m-u}}{1-\alpha_j}\end{bmatrix}.$$

Then, the equilibrium $\mathcal{H}$ satisfies

$$e_j'\mathcal{H}'=e_j'\mathcal{A}\Omega_{\Gamma_j}B'\left(B\Omega_{\Gamma_j}B'\right)^{-1}.$$

Proof. It follows from Lemma B.1 that

$$\mathcal{H}'=\left(\mathrm{I}-\Phi\right)\mathcal{A}\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}+\Phi\mathcal{H}'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}.$$

Right multiplying by $B\hat{\Omega}B'$, we obtain

$$\mathcal{H}'B\hat{\Omega}B'=\left(\mathrm{I}-\Phi\right)\mathcal{A}\hat{\Omega}B'+\Phi\mathcal{H}'B\Lambda\hat{\Omega}B',$$

or, using $\Phi=\sum_{j=1}^{J}e_je_j'\alpha_j$,

$$\sum_{j=1}^{J}e_je_j'\mathcal{H}'B\hat{\Omega}B'-\sum_{j=1}^{J}\alpha_je_je_j'\mathcal{H}'B\Lambda\hat{\Omega}B'=\left(\mathrm{I}-\Phi\right)\mathcal{A}\hat{\Omega}B',$$

which can be rewritten as

$$\sum_{j=1}^{n}e_je_j'\mathcal{H}'B\left(\mathrm{I}-\alpha_j\Lambda\right)\hat{\Omega}B'=\left(\mathrm{I}-\Phi\right)\mathcal{A}\hat{\Omega}B'.$$

Since $(\mathrm{I}-\alpha_j\Lambda)=(1-\alpha_j)\Gamma_j$, it follows that

$$\sum_{j=1}^{n}\left(1-\alpha_j\right)e_je_j'\mathcal{H}'B\Gamma_j\hat{\Omega}B'=\left(\mathrm{I}-\Phi\right)\mathcal{A}\hat{\Omega}B'.$$

Guessing that

$$e_j'\mathcal{H}'=e_j'\mathcal{A}\Omega_{\Gamma_j}B'\left(B\Omega_{\Gamma_j}B'\right)^{-1},$$

and using $A\Gamma_j = A$, we obtain

$$\sum_{j=1}^{n} (1-\alpha_j)\, e_j e_j' \mathcal{A}\hat{\Omega}B' \left(B\Omega_{\Gamma_j}B'\right)^{-1} B\Omega_{\Gamma_j}B' = (\mathrm{I}-\Phi)\,\mathcal{A}\hat{\Omega}B',$$

or

$$\sum_{j=1}^{n} (1-\alpha_j)\, e_j e_j' \mathcal{A}\hat{\Omega}B' = (\mathrm{I}-\Phi)\,\mathcal{A}\hat{\Omega}B'.$$

The fact that $(\mathrm{I}-\Phi) = \sum_{j=1}^{J} (1-\alpha_j)\, e_j e_j'$ concludes the proof. □

Lemma B.3. *Define*

$$\Delta_j \equiv \Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}, \quad \textit{and} \quad \tilde{\Omega}_\mu \equiv \left(\Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1},$$

with

$$\hat{W} \equiv \mathrm{I}_u - \lambda\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\bar{W}\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}',$$

and

$$\bar{W} \equiv \left(\mathrm{I}_J + \lambda\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\right)^{-1}.$$

Then, the equilibrium $\mathcal{H}$ satisfies

$$\mathcal{H}' = \sum_{i=1}^{J} e_j e_j' \mathcal{A}\Delta_j B' \left(B\Delta_j B'\right)^{-1}.$$

Proof. It follows from Lemma B.2 that

$$\sum_{j=1}^{n} e_j e_j' \left(\mathcal{A}-\mathcal{H}'B\right)\Omega_{\Gamma_j}B' = 0.$$

From the definitions of Ω_{Γ_j} and $\tilde{\Omega}_\mu$, we have that

$$\Omega_{\Gamma_j} = \Gamma_j\Omega + \mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} + \lambda\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}.$$

It is then sufficient to show that

$$\begin{aligned}
&\sum_{j=1}^{n} e_j e_j' \left(\mathcal{A}-\mathcal{H}'B\right)\left(\Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right) \\
&= \sum_{j=1}^{n} e_j e_j' \left(\mathcal{A}-\mathcal{H}'B\right)\left(\Gamma_j\Omega + \mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} + \lambda\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}\right),
\end{aligned}$$

or, equivalently,

$$\left(\mathcal{A}-\mathcal{H}'B\right)\left(\mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right) = \left(\mathcal{A}-\mathcal{H}'B\right)\left(\mathcal{K}'\left(\tilde{\Omega}_\mu^{-1} + \lambda\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}\right).$$

In turn, a sufficient condition for this equation to be satisfied is that

$$\hat{W} = \left(\mathrm{I}_u + \lambda\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A} - \mathcal{H}'B\right)'\hat{\Psi}_k\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\right)^{-1},$$

which, using the Woodbury matrix identity, can be rewritten as

$$\hat{W} = \mathrm{I}_u - \lambda\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A} - \mathcal{H}'B\right)'\hat{\Psi}_k\bar{W}\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}',$$

with

$$\bar{W} = \left(\mathrm{I}_J + \lambda\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A} - \mathcal{H}'B\right)'\hat{\Psi}_k\right)^{-1}.$$

□

Lemma B.4. *Denote the eigenvalue decomposition of* $(\mathrm{I}_J - \Phi)\left(\tau_\mu^{-1}\mathrm{I}_J + \bar{W}\right)$ *by*

$$(\mathrm{I}_J - \Phi)\left(\tau_\mu^{-1}\mathrm{I}_J + \bar{W}\right) = P^{-1}\left(\sum_{j=1}^{J}\omega_j e_j e_j'\right)P.$$

Define

$$\bar{\Delta}_j \equiv \Gamma_j\Omega + \mathcal{K}'\left(\frac{\omega_j}{(1-\alpha_j)}\Omega_\mu - \Omega_\eta\right)\mathcal{K},$$

and let the scalars $\hat{r}_j$ *and* $\hat{x}_j$ *be given by*

$$\hat{r}_j \equiv \frac{\lambda\gamma e_j'\bar{W}\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A'}{(1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A')\hat{\kappa}_j}, \quad \text{and} \quad \hat{x}_j \equiv \sum_{i=1}^{J}P_{ji}\left(1 + \frac{(1-\alpha_i)\,\hat{r}_i}{\omega_j}\right)\hat{\kappa}_i,$$

and let $\mathcal{X}'$ *be such that*

$$e_j'\mathcal{X}' \equiv \hat{x}_j\left(A\bar{\Delta}_jB'\right)\left(B\bar{\Delta}_jB'\right)^{-1}.$$

Then, the equilibrium $\mathcal{H}$ *satisfies*

$$\mathcal{H}' = P^{-1}\mathcal{X}'.$$

Proof. From Lemma B.3, we have that

$$e_j'\mathcal{H}' = e_j'\mathcal{A}\left(\Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right)B'\left(B\left(\Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right)B'\right)^{-1},$$

and, therefore,

$$e_j'\mathcal{H}'B\left(\Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right)B' = e_j'\mathcal{A}\left(\Gamma_j\Omega + \mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}\right)B'.$$

Rearranging, we get

$$e_j'\mathcal{H}'B\Gamma_j\Omega B' = e_j'\mathcal{A}\Gamma_j\Omega B' + e_j'\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\hat{W}\tilde{\Omega}_\mu\mathcal{K}B'.$$

Since

$$\begin{aligned}(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\hat{W} &= (\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}' - \lambda\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\bar{W}\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}' \\ &= \left(\mathrm{I}_J - \lambda\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'\hat{\Psi}_k\right)\bar{W}\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}' \\ &= \bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}',\end{aligned}$$

it follows that

$$e_j'\mathcal{H}'B\Gamma_j\Omega B' = e_j'\mathcal{A}\Gamma_j\Omega B' + e_j'\bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}B'.$$

From the definition of $\tilde{\Omega}_\mu$, we have that

$$\tilde{\Omega}_\mu \equiv \left(\Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1} = \Omega_\mu + s\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu,$$

with

$$s \equiv \frac{\lambda\gamma}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}.$$

So that

$$\begin{aligned}e_j'\mathcal{H}'B\Gamma_j\Omega B' &= e_j'\mathcal{A}\Gamma_j\Omega B' + e_j'\bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\left(\Omega_\mu + s\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}B' \\ &= e_j'\mathcal{A}\Gamma_j\Omega B' + e_j'\bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}B' + \hat{r}_j e_j'\mathcal{A}\mathcal{K}'\Omega_\mu\mathcal{K}B',\end{aligned}$$

where we used the fact that $e_j'\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A'$ and $\hat{\kappa}_j \equiv e_j'\hat{\kappa}$ are scalars, $\hat{\kappa}_j A = e_j'\mathcal{A}$, and

$$\hat{r}_j \equiv \frac{s\left(e_j'\bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A'\right)}{\hat{\kappa}_j}.$$

Thus, it follows that

$$e_j'\mathcal{H}'B\Gamma_j\Omega B' = e_j'\mathcal{A}\left(\Gamma_j\Omega + \hat{r}_j\mathcal{K}'\Omega_\mu\mathcal{K}\right)B' + e_j'\bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}B',$$

which implies

$$\sum_{j=1}^{J} e_j e_j'\mathcal{H}'B\Gamma_j\Omega B' = \sum_{j=1}^{J} e_j e_j'\mathcal{A}\left(\Gamma_j\Omega + \hat{r}_j\mathcal{K}'\Omega_\mu\mathcal{K}\right)B' + \bar{W}\,(\mathcal{A}-\mathcal{H}'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}B',$$

and, therefore,

$$\sum_{j=1}^{J} e_j e_j'\mathcal{H}'B\Gamma_j\Omega B' + \bar{W}\mathcal{H}'B\mathcal{K}'\Omega_\mu\mathcal{K}B' = \sum_{j=1}^{J} e_j e_j'\mathcal{A}\left(\Gamma_j\Omega + \hat{r}_j\mathcal{K}'\Omega_\mu\mathcal{K}\right)B' + \bar{W}\mathcal{A}\mathcal{K}'\Omega_\mu\mathcal{K}B'.$$

Next, we use this equation to solve for $\mathcal{H}$. Recall that $A = [a', 0]$, where a is of dimension $u \times 1$, and let

$B = [B_1, B_2]$, with B_1 of dimension $n \times u$ and B_2 of dimension $n \times (m-u)$. Then,

$$\sum_{j=1}^{J} e_j e_j' \mathcal{H}' \left(B_1 \Omega_\eta B_1' + \frac{1}{1-\alpha_j} B_2 \Omega_\varepsilon B_2' \right) + \bar{W} \mathcal{H}' \left(B_1 \Omega_\mu B_1' \right) = \sum_{j=1}^{J} e_j \hat{\kappa}_j a' \left(\Omega_\eta + \hat{r}_j \Omega_\mu \right) B_1' + \bar{W} \hat{\kappa} a' \Omega_\mu B_1'.$$

Using the fact that $\tau_\mu^{-1} \Omega_\mu = \Omega_\eta$, it follows that

$$\left(\tau_\mu^{-1} \mathrm{I}_J + \bar{W} \right) \mathcal{H}' \left(B_1 \Omega_\mu B_1' \right) + \left(\mathrm{I}_J - \Phi \right)^{-1} \mathcal{H}' \left(B_2 \Omega_\varepsilon B_2' \right) = \sum_{j=1}^{J} e_j \hat{\kappa}_j a' \left(\hat{r}_j \Omega_\mu \right) B_1' + \left(\tau_\mu^{-1} \mathrm{I}_J + \bar{W} \right) \hat{\kappa} a' \Omega_\mu B_1'.$$

Left multiplying by $(\mathrm{I}_J - \Phi)$, then, implies

$$\left(\mathrm{I}_J - \Phi \right) \left(\tau_\mu^{-1} \mathrm{I}_J + \bar{W} \right) \mathcal{H}' \left(B_1 \Omega_\mu B_1' \right) + \mathcal{H}' \left(B_2 \Omega_\varepsilon B_2' \right) = \left(\mathrm{I}_J - \Phi \right) \left(\sum_{j=1}^{J} e_j \hat{\kappa}_j a' \left(\hat{r}_j \Omega_\mu \right) B_1' + \left(\tau_\mu^{-1} \mathrm{I}_J + \bar{W} \right) \hat{\kappa} a' \Omega_\mu B_1' \right).$$

Since, by definition,

$$\left(\mathrm{I}_J - \Phi \right) \left(\tau_\mu^{-1} \mathrm{I}_J + \bar{W} \right) = P^{-1} D P, \quad \text{with} \quad D \equiv \left(\sum_{j=1}^{J} \omega_j e_j e_j' \right),$$

it follows that

$$P^{-1} D P \mathcal{H}' \left(B_1 \Omega_\mu B_1' \right) + \mathcal{H}' \left(B_2 \Omega_\varepsilon B_2' \right) = \left(\mathrm{I}_J - \Phi \right) \sum_{j=1}^{J} e_j \hat{\kappa}_j a' \left(\hat{r}_j \Omega_\mu \right) B_1' + P^{-1} D P \hat{\kappa} a' \Omega_\mu B_1'.$$

Left multiplying by P, then, implies

$$D P \mathcal{H}' \left(B_1 \Omega_\mu B_1' \right) + P \mathcal{H}' \left(B_2 \Omega_\varepsilon B_2' \right) = P \left(\mathrm{I}_J - \Phi \right) \sum_{j=1}^{J} e_j \hat{\kappa}_j a' \left(\hat{r}_j \Omega_\mu \right) B_1' + D P \hat{\kappa} a' \Omega_\mu B_1'.$$

Next, define

$$\mathcal{X}' \equiv P \mathcal{H}',$$

so that we can rewrite the equation as

$$D \mathcal{X}' \left(B_1 \Omega_\mu B_1' \right) + \mathcal{X}' \left(B_2 \Omega_\varepsilon B_2' \right) = P \sum_{j=1}^{J} e_j \left(1 - \alpha_j \right) \hat{\kappa}_j a' \left(\hat{r}_j \Omega_\mu \right) B_1' + D P \hat{\kappa} a' \Omega_\mu B_1'.$$

Next, using the definition of D, we obtain

$$e_j' \mathcal{X}' \left(B_1 \omega_j \Omega_\mu B_1' + B_2 \Omega_\varepsilon B_2' \right) = e_j' \left(P \sum_{i=1}^{J} e_i \left(1 - \alpha_i \right) \hat{\kappa}_i a' \left(\hat{r}_i \Omega_\mu \right) B_1' + D P \hat{\kappa} a' \Omega_\mu B_1' \right).$$

Right multiplying by $(B_1\omega_j\Omega_\mu B_1' + B_2\Omega_\varepsilon B_2')^{-1}$, then, yields

$$e_j'\mathcal{X}' = e_j'\left(P\sum_{i=1}^{J} e_i\left(1-\alpha_i\right)\hat{\kappa}_i a'\left(\hat{r}_i\Omega_\mu\right)B_1' + DP\hat{\kappa}a'\Omega_\mu B_1'\right)\left(B_1\omega_j\Omega_\mu B_1' + B_2\Omega_\varepsilon B_2'\right)^{-1}.$$

Notice that

$$\begin{aligned} e_j'\left(P\sum_{i=1}^{J} e_i\left(1-\alpha_i\right)\hat{\kappa}_i a'\left(\hat{r}_i\Omega_\mu\right)B_1' + DP\hat{\kappa}a'\Omega_\mu B_1'\right) &= \sum_{i=1}^{J} e_j' P e_i\left(1-\alpha_i\right)\hat{\kappa}_i a'\left(\hat{r}_i\Omega_\mu\right)B_1' + e_j' DP\hat{\kappa}a'\Omega_\mu B_1' \\ &= \sum_{i=1}^{J} P_{ji}\left(\left(1-\alpha_i\right)\hat{r}_i + \omega_j\right)\hat{\kappa}_i a'\Omega_\mu B_1', \end{aligned}$$

so that we can further rewrite the expression as

$$\begin{aligned} e_j'\mathcal{X}' &= \left(\sum_{i=1}^{J} P_{ji}\left(\frac{\left(1-\alpha_i\right)\hat{r}_i + \omega_j}{\omega_j}\right)\hat{\kappa}_i a'\omega_j\Omega_\mu B_1'\right)\left(B_1\omega_j\Omega_\mu B_1' + B_2\Omega_\varepsilon B_2'\right)^{-1} \\ &= \left(\sum_{i=1}^{J} P_{ji}\left(\frac{\left(1-\alpha_i\right)\hat{r}_i + \omega_j}{\omega_j}\right)\hat{\kappa}_i a'\frac{\omega_j}{\left(1-\alpha_j\right)}\Omega_\mu B_1'\right)\left(B_1\frac{\omega_j}{\left(1-\alpha_j\right)}\Omega_\mu B_1' + B_2\frac{1}{\left(1-\alpha_j\right)}\Omega_\varepsilon B_2'\right)^{-1}. \end{aligned}$$

Finally, using the definition of $\bar{\Delta}_j$, we get

$$e_j'\mathcal{X}' = \sum_{i=1}^{J} P_{ji}\left(1 + \frac{\left(1-\alpha_i\right)\hat{r}_i}{\omega_j}\right)\hat{\kappa}_i\left(A\bar{\Delta}_j B'\right)\left(B\bar{\Delta}_j B'\right)^{-1},$$

and the definition of $\mathcal{X}'$ implies

$$\mathcal{H}' = P^{-1}\mathcal{X}'.$$

□

Parts 1 and 2 of Proposition 3.6. Given the result in Lemma B.4, we are left with taking the limit, as $T\to\infty$, of the truncated problem. Define $w_j \equiv \frac{\tau_\mu\omega_j}{1-\alpha_j} - 1$, then, in particular, we have that

$$\lim_{T\to\infty} A\bar{\Delta}_j B'\left(B\bar{\Delta}_j B'\right)^{-1} = p\left(L; w_j, \alpha_j\right), \quad \lim_{T\to\infty} A\mathcal{K}'\Omega_\eta\mathcal{K}A' = \mathbb{V}(\xi_t),$$

$$\lim_{T\to\infty}\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}\left(\mathcal{A} - \mathcal{H}'B\right)' = \mathbb{V}(\hat{\kappa}\xi_t - \hat{K}_t), \quad \lim_{T\to\infty}\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\Omega_\mu\mathcal{K}A' = \mathbb{COV}\left(\hat{\kappa}\xi_t - \hat{K}_t, \xi_t\right).$$

Next, let $W \equiv \lim_{T\to\infty} \tau_\mu \bar{W}$, $r_j \equiv \lim_{T\to\infty} \frac{\tau_\mu}{1+w_j} \hat{r}_j$, and $x_j \equiv \lim_{T\to\infty} \hat{x}_j$, for $j \in \{1, \ldots, J\}$. Then, it follows that

$$
\begin{aligned}
W &= \lim_{T\to\infty} \left(\tau_\mu^{-1} \mathrm{I}_J + \lambda \tau_\mu^{-1} \left(\mathcal{A} - \mathcal{H}'B\right) \mathcal{K}' \left(\Omega_\mu + \frac{\lambda\gamma\Omega_\mu \mathcal{K} A' A \mathcal{K}' \Omega_\mu}{1 - \lambda\gamma A \mathcal{K}' \Omega_\mu \mathcal{K} A'} \right) \mathcal{K} \left(\mathcal{A} - \mathcal{H}'B\right)' \hat{\Psi}_k \right)^{-1} \\
&= \lim_{T\to\infty} \left(\tau_\mu^{-1} \mathrm{I}_J + \lambda \left(\left(\mathcal{A} - \mathcal{H}'B\right) \mathcal{K}' \Omega_\eta \mathcal{K} \left(\mathcal{A} - \mathcal{H}'B\right)' + \frac{\lambda\gamma\left(\mathcal{A}-\mathcal{H}'B\right)\mathcal{K}'\left(\Omega_\eta \mathcal{K} A' A \mathcal{K}' \Omega_\eta\right)\mathcal{K}\left(\mathcal{A}-\mathcal{H}'B\right)'}{\tau_\mu^{-1} - \lambda\gamma A \mathcal{K}' \Omega_\eta \mathcal{K} A'} \right) \hat{\Psi}_k \right)^{-1} \\
&= \left(\tau_\mu^{-1} \mathrm{I}_J + \lambda \left(\mathbb{V}\left(\hat{\kappa}_1 \xi_t - \hat{K}_t\right) + \frac{\lambda\gamma \mathbb{COV}\left(\hat{\kappa}\xi_t - \hat{K}_t, \xi_t\right) \mathbb{COV}\left(\hat{\kappa}\xi_t - \hat{K}_t, \xi_t\right)'}{\tau_\mu^{-1} - \lambda\gamma \mathbb{V}\left(\xi_t\right)} \right) \hat{\Psi}_k \right)^{-1} \\
W &= \left(\tau_\mu^{-1} \mathrm{I}_J + \lambda Q \left(\mathbb{V}\left(\kappa\xi_t - K_t\right) + \frac{\lambda\gamma \mathbb{COV}\left(\kappa\xi_t - K_t, \xi_t\right) \mathbb{COV}\left(\kappa\xi_t - K_t, \xi_t\right)'}{\tau_\mu^{-1} - \lambda\gamma \mathbb{V}\left(\xi_t\right)} \right) \Psi_k Q^{-1} \right)^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
r_j &= \lim_{T\to\infty} \frac{\tau_\mu}{1+w_j} \frac{\lambda\gamma}{1 - \lambda\gamma A \mathcal{K}' \Omega_\mu \mathcal{K} A'} \frac{e_j' \tau_\mu^{-1} \tau_\mu \bar{W} \left(\mathcal{A} - \mathcal{H}'B\right) \mathcal{K}' \Omega_\mu \mathcal{K} A'}{\hat{\kappa}_j} \\
&= \frac{\tau_\mu}{1+w_j} \frac{\lambda\gamma}{1 - \lambda\tau_\mu\gamma\mathbb{V}\left(\xi_t\right)} \frac{e_j' W \mathbb{COV}\left(\hat{\kappa}\xi_t - \hat{K}_t, \xi_t\right)}{\hat{\kappa}_j} \\
&= \gamma \frac{\lambda\tau_\mu}{1 - \lambda\tau_\mu\gamma\mathbb{V}\left(\xi_t\right)} \frac{e_j' W Q \mathbb{COV}\left(\kappa\xi_t - K_t, \xi_t\right)}{e_j' Q \kappa_j (1 + w_j)},
\end{aligned}
$$

and

$$
x_j = \lim_{T\to\infty} \sum_{i=1}^{J} P_{ji} \left(1 + \frac{\left(1-\alpha_i\right)\hat{r}_i}{\omega_j} \right) \hat{\kappa}_i \; = \; \sum_{i=1}^{J} P_{ji} \left(1 + \frac{\left(1-\alpha_i\right)}{\left(1-\alpha_j\right)} r_j \right) Q\kappa_i.
$$

Part 3 of Proposition 3.6. Next, we characterize the bias term, $\mathcal{B} \equiv \lim_{T\to\infty} Q^{-1}\mathcal{Q}$. From equation (B.6), we have that

$$
\left(\mathrm{I}_J - \Phi\right) \mathcal{Q} = \left[\left(\mathrm{I} - \Phi\right)\left(\mathcal{A} - \mathrm{T}B\right) + \Phi\mathcal{H}'\left(B\Lambda - \mathrm{H}B\right) \right] \mathcal{K}' S \left(-\lambda\mathcal{K}\left(\mathcal{H}'B - \mathcal{A}\right)' \hat{\Psi}_k \mathcal{Q} - \lambda\chi\mathcal{K}A' \right).
$$

There exists a unique $\mathcal{Q}$ that satisfies this equation. We postulate that there exists $\mathcal{Y}$ such that

$$
\left(\mathrm{I}_J - \Phi\right) \mathcal{Q} = \left[\left(\mathrm{I}_J - \Phi\right)\mathcal{A} + \Phi\mathcal{H}'B\Lambda - \mathcal{H}'B \right] \mathcal{K}'\mathcal{Y},
$$

so that solving for $\mathcal{Y}$ pins down the unique $\mathcal{Q}$. To proceed, first replace the guess for $\mathcal{Q}$ on the RHS of equation (B.6),

$$
\begin{aligned}
\mathrm{RHS} \equiv \; & \left[\left(\mathrm{I}_J - \Phi\right)\left(\mathcal{A} - \mathrm{T}B\right) + \Phi\mathcal{H}'\left(B\Lambda - \mathrm{H}B\right) \right] \mathcal{K}' S \left(-\lambda\mathcal{K}\left(\mathcal{H}'B - \mathcal{A}\right)' \hat{\Psi}_k \left(\mathrm{I}_J - \Phi\right)^{-1} \left(\mathrm{I}_J - \Phi\right) \mathcal{Q} - \lambda\chi\mathcal{K}A' \right) \\
= \; & \left[\left(\mathrm{I}_J - \Phi\right)\left(\mathcal{A} - \mathrm{T}B\right) + \Phi\mathcal{H}'\left(B\Lambda - \mathrm{H}B\right) \right] \mathcal{K}' S \\
& \times \left(-\lambda\mathcal{K}\left(\mathcal{H}'B - \mathcal{A}\right)' \hat{\Psi}_k \left(\mathrm{I}_J - \Phi\right)^{-1} \left[\left(\mathrm{I}_J - \Phi\right)\mathcal{A} + \Phi\mathcal{H}'B\Lambda - \mathcal{H}'B \right] \mathcal{K}'\mathcal{Y} - \lambda\chi\mathcal{K}A' \right).
\end{aligned}
$$

Next, for the LHS of the equation, we have that

$$\text{LHS} \equiv (\mathrm{I}_J - \Phi)\,\mathcal{Q} = [(\mathrm{I}_J - \Phi)\,\mathcal{A} + \Phi\mathcal{H}' B\Lambda - \mathcal{H}' B]\,\mathcal{K}'\mathcal{Y},$$

and, substituting the last $\mathcal{H}'$ using equation (B.5), it follows that

$$\begin{aligned}
\text{LHS} =&\ [(\mathrm{I}_J - \Phi)(\mathcal{A} - \mathcal{T}B) + \Phi\mathcal{H}'(B\Lambda - \mathrm{H}B)]\left[\mathrm{I}_m - \mathcal{K}' S\mathcal{K}B'(B\Omega B')^{-1} B\right]\mathcal{K}'\mathcal{Y} \\
=&\ [(\mathrm{I}_J - \Phi)(\mathcal{A} - \mathrm{T}B) + \Phi\mathcal{H}'(B\Lambda - \mathrm{H}B)]\,\mathcal{K}' S\left[S^{-1} - \mathcal{K}B'(B\Omega B')^{-1} B\mathcal{K}'\right]\mathcal{Y} \\
=&\ [(\mathrm{I}_J - \Phi)(\mathcal{A} - \mathrm{T}B) + \Phi\mathcal{H}'(B\Lambda - \mathrm{H}B)]\,\mathcal{K}' S \\
&\times\left\{\Omega_\mu^{-1} + \lambda\left(\mathcal{K}(\mathcal{H}'B - \mathcal{A})'\,\hat{\Psi}_k\,(\mathcal{H}'B - \mathcal{A})\,\mathcal{K}' - \gamma\mathcal{K}A'A\mathcal{K}'\right)\right\}\mathcal{Y},
\end{aligned}$$

where the last equality uses the definition of S. Putting these results together, we have that

$$\begin{aligned}
\text{LHS} - \text{RHS} =&\ [(\mathrm{I}_J - \Phi)(\mathcal{A} - \mathrm{T}B) + \Phi\mathcal{H}'(B\Lambda - \mathrm{H}B)]\,\mathcal{K}' S \\
&\times\left\{\begin{array}{c}\lambda\mathcal{K}(\mathcal{H}'B - \mathcal{A})'\,\hat{\Psi}_k\left\{(\mathcal{H}'B - \mathcal{A})\,\mathcal{K}' + (\mathrm{I}_J - \Phi)^{-1}[(\mathrm{I}_J - \Phi)\,\mathcal{A} + \Phi\mathcal{H}'B\Lambda - \mathcal{H}'B]\,\mathcal{K}'\right\}\mathcal{Y} \\ \Omega_\mu^{-1}\mathcal{Y} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\mathcal{Y} - \lambda\chi\mathcal{K}A'\end{array}\right\} \\
=&\ [(\mathrm{I}_J - \Phi)(\mathcal{A} - \mathrm{T}B) + \Phi\mathcal{H}'(B\Lambda - \mathrm{H}B)]\,\mathcal{K}' S \\
&\times\left\{\Omega_\mu^{-1}\mathcal{Y} + \lambda\mathcal{K}(\mathcal{H}'B - \mathcal{A})'\,\hat{\Psi}_k\,(\mathrm{I}_J - \Phi)^{-1}\,\Phi\mathcal{H}'B\,(\Lambda - \mathrm{I}_m)\,\mathcal{K}'\mathcal{Y} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\mathcal{Y} - \lambda\chi\mathcal{K}A'\right\}.
\end{aligned}$$

Since $(\Lambda - \mathrm{I}_m)\,\mathcal{K}' = 0$, a sufficient condition for $\text{LHS} - \text{RHS} = 0$ is

$$\Omega_\mu^{-1}\mathcal{Y} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\mathcal{Y} - \lambda\chi\mathcal{K}A' = 0,$$

which, using the Sherman-Morrison formula, implies that

$$\mathcal{Y} = \lambda\chi\left(\Omega_\mu^{-1} - \lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1}\mathcal{K}A' = \lambda\chi\left(\Omega_\mu + \frac{\lambda\gamma}{1 - \lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}A'.$$

Therefore, we have that

$$\begin{aligned}
\mathcal{Q} =&\ (\mathrm{I}_J - \Phi)^{-1}[(\mathrm{I}_J - \Phi)\,\mathcal{A} + \Phi\mathcal{H}'B\Lambda - \mathcal{H}'B]\,\mathcal{K}'\mathcal{Y} \\
=&\ (\mathrm{I}_J - \Phi)^{-1}\left[(\mathrm{I}_J - \Phi)\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}' + \Phi\mathcal{H}'B\,(\Lambda - \mathrm{I}_m)\,\mathcal{K}'\right]\mathcal{Y} \\
=&\ \left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\mathcal{Y} \\
=&\ \left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\lambda\chi\left(\Omega_\mu + \frac{\lambda\gamma}{1 - \lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\Omega_\mu\mathcal{K}A'A\mathcal{K}'\Omega_\mu\right)\mathcal{K}A' \\
=&\ \lambda\tau_\mu\chi\left(1 + \frac{\lambda\gamma\tau_\mu A\mathcal{K}'\Omega_\eta\mathcal{K}A'}{1 - \lambda\gamma\tau_\mu A\mathcal{K}'\Omega_\eta\mathcal{K}A'}\right)\left(\mathcal{A} - \mathcal{H}'B\right)\mathcal{K}'\Omega_\eta\mathcal{K}A'.
\end{aligned}$$

Taking the limit we obtain

$$\mathcal{B} = \lim_{T\to\infty} Q^{-1}\mathcal{Q} = \frac{\lambda\tau_\mu\chi}{1 - \lambda\gamma\tau_\mu\mathbb{V}(\xi_t)}\mathbb{COV}(\kappa\xi_t - K_t, \xi_t),$$

which completes the proof of part 3 of the proposition.

B.2 Inefficient economies

The economy in our baseline setup is assumed to be efficient under both complete and incomplete information. We now consider a generalized utility in the vein of Angeletos and Pavan (2007),

$$u\left(k_{it}, k_t, \xi_t\right) = -\frac{1}{2}\left[\left(1-\alpha\right)\left(k_{it}-\xi_t\right)^2 + \alpha\left(k_{it}-K_t\right)^2\right] - \frac{1}{2}\gamma\xi_t^2 - \chi\xi_t - \frac{1}{2}\psi\left(K_t-\xi_t\right)^2 - \phi K_t\xi_t - \varphi K_t, \quad \text{(B.8)}$$

which allows inefficiencies under both complete and incomplete information. Specifically, it can be shown that:

- Under complete information, the equilibrium allocation is such that $k_{it} = K_t = \xi_t$, whereas the efficient allocation is such that $k_{it} = K_t = \kappa_1^*\xi_t + \kappa_0^*$ with (κ_1^*, κ_0^*) being given by

$$\kappa_1^* = \frac{1-(\alpha-\psi)-\phi}{1-(\alpha-\psi)}, \quad \text{, and} \quad \kappa_0^* = \frac{\varphi}{1-(\alpha-\psi)}.$$

- Under incomplete information, the equilibrium degree of coordination is α, while the efficient degree of coordination is $\alpha^* = \alpha - \psi$.

The following proposition generalizes our equivalence result to the utility function in equation (B.8). The equilibrium strategy still features the simple form, which results in additional sensitivity and bias.

Proposition B.3. *The linear strategy in equilibrium takes the following form*

$$g(x_i^t) = (1+r)p(L; w, \alpha)x_{it} + \mathcal{B}. \quad \text{(B.9)}$$

1. *The polynomial matrix $p(L; w, \alpha)$ is the Bayesian forecasting rule with the (w, α)-modified signal process and w satisfies*

$$w = \frac{\tau_\mu}{\left(1+\nu_1\right) - \lambda\left(1-\alpha+\psi\right)\tau_\mu\left(\mathbb{V}\left(\xi_t - K_t\right) + \frac{\lambda\gamma\tau_\mu(1+\nu_2)\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2}{1-\lambda\gamma\tau_\mu(1+\nu_3)\mathbb{V}(\xi_t)}\right)};$$

2. *The additional amplification, r, satisfies*

$$r = \frac{\gamma\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)\left(1+\nu_2\right)}{1-\gamma\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)\left(1+\nu_3\right)}\frac{w}{1+w}\left(1-\mathcal{S}\right);$$

3. *The level of bias, $\mathcal{B}$, satisfies*

$$\mathcal{B} = \frac{\chi\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)\left(1-\mathcal{S}\right)+\nu_4}{1-\gamma\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)+\nu_5};$$

4. Relative to Proposition 3.3, the inefficiencies imply the following correction terms

$$\nu_1 \equiv \frac{\lambda^2\phi^2\tau_\mu^2\left(\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2 - \mathbb{V}(\xi_t)\mathbb{V}(\xi_t - K_t)\right) - \lambda\phi\tau_\mu\mathbb{V}(\xi_t)(1-\mathcal{S})}{1-\lambda\tau_\mu\mathbb{V}(\xi_t)(2\gamma - \phi(1+\mathcal{S}))},$$

$$\nu_2 \equiv 1 - \frac{\phi}{\gamma}\left(2 - \frac{\mathbb{V}(\xi_t - K_t)}{\mathbb{V}(\xi_t)(1-\mathcal{S})}\right),$$

$$\nu_3 \equiv 1 - \frac{\phi}{\gamma}(1+\mathcal{S}),$$

$$\nu_4 \equiv \lambda\varphi\tau_\mu(\mathbb{V}(\xi_t)(1-\mathcal{S}) - \mathbb{V}(\xi_t - K_t)) - \lambda^2\tau_\mu^2(\phi(\chi-\varphi)+2\gamma\varphi)\left(\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2 - \mathbb{V}(\xi_t)\mathbb{V}(\xi_t-K_t)\right)$$

$$\nu_5 \equiv \lambda\tau_\mu\mathbb{V}(\xi_t)(2\phi\mathcal{S} - \gamma) + \lambda^2\tau_\mu^2\phi^2\left(\mathbb{V}(\xi_t)^2(1-\mathcal{S})^2 - \mathbb{V}(\xi_t)\mathbb{V}(\xi_t-K_t)\right).$$

It is easy to see that without inefficiencies, that is if $\psi = \phi = \varphi = 0$, we have that $\nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu_5 = 0$, and the formulas reduce to the ones in Proposition 3.3.

Proof of Proposition B.3. Consider the same truncated version of the model described in the proof of Proposition 3.3. For the utility in equation (B.8), we have that

$$\hat{p}(\mu|x_i) \propto \exp\left(-\frac{1}{2}\mu' S^{-1}\mu + \frac{1}{2}\mu' S^{-1}(Mx_i + \pi) + \frac{1}{2}(Mx_i+\pi)' S^{-1}\mu\right),$$

where matrices M, π, and S are such that

$$M \equiv S\mathcal{K}(B\Omega B')^{-1}, \quad \pi \equiv S\left[-\lambda(1-\alpha^*)h_0\mathcal{K}(A'-B'h) + \lambda\chi\mathcal{K}A' + \lambda\varphi\mathcal{K}B'h\right],$$

and

$$S \equiv \left(\mathcal{K}B'(B\Omega B')^{-1}B\mathcal{K}' + \Omega_\mu^{-1} - \lambda\left[(1-\alpha^*)\mathcal{K}(A'-B'h)(A-h'B)\mathcal{K}' + \gamma\mathcal{K}A'A\mathcal{K}'\right] - \lambda\phi\mathcal{K}(\Lambda B'hA + A'h'B\Lambda)\mathcal{K}'\right)^{-1},$$

which, using $\phi = (1-\alpha^*)(1-\kappa_1^*)$, can be rearranged into

$$S = \left(\mathcal{K}B'(B\Omega B')^{-1}B\mathcal{K}' + \Omega_\mu^{-1} - \lambda\gamma^*\mathcal{K}A'A\mathcal{K}' - \lambda(1-\alpha^*)\mathcal{K}(\kappa_1^*A' - B'h)(\kappa_1^*A - h'B)\mathcal{K}'\right)^{-1},$$

where

$$\gamma^* \equiv \gamma + (1-\alpha^*)(1-(\kappa_1^*)^2).$$

We have the same equilibrium conditions for h and h_0 as in Proposition 3.3, equations (A.4) and (A.5), and the proof proceeds analogously and we keep the same structure to facilitate comparison.

Lemma B.5. *Define*

$$\hat{\Omega} \equiv \Omega + \mathcal{K}'W\mathcal{K}, \quad \hat{\mathrm{T}} \equiv A\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}, \quad \hat{\mathrm{H}} \equiv B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1},$$

and

$$W \equiv \left(\Omega_\mu^{-1} - \lambda\gamma^* \mathcal{K} A' A \mathcal{K}' - \lambda\left(1-\alpha^*\right) \mathcal{K}\left(\kappa_1^* A' - B'h\right)\left(\kappa_1^* A - h'B\right) \mathcal{K}'\right)^{-1}.$$

Then, the equilibrium h solves the following fixed-point problem

$$h' = (1-\alpha)\,\hat{\mathrm{T}} + \alpha h' \hat{\mathrm{H}}.$$

Proof. This proof is exactly analogous to the proof of Lemma A.1. In particular, notice that W and S are still such that

$$S = \left(\mathcal{K} B' \left(B \Omega B'\right)^{-1} B \mathcal{K}' + W^{-1}\right)^{-1}.$$

□

Lemma B.6. *Define*

$$\Omega_\Gamma \equiv \Gamma\,\hat{\Omega}, \quad \textit{with} \quad \Gamma \equiv \begin{bmatrix} \mathrm{I}_u & 0_{u,m-u} \\ 0_{m-u,u} & \frac{\mathrm{I}_{m-u}}{1-\alpha} \end{bmatrix}.$$

Then, the equilibrium h satisfies

$$h' = A \Omega_\Gamma B' \left(B \Omega_\Gamma B'\right)^{-1}.$$

Proof. This lemma is exactly the same as Lemma A.2, and is repeated here just for convenience. □

Lemma B.7. *Define*

$$\Delta \equiv \Gamma\Omega + \hat{w}\tau_\mu^{-1} \mathcal{K}' \tilde{\Omega}_\mu \mathcal{K},$$

and

$$\tilde{\Omega}_\mu \equiv \left(\Omega_\mu^{-1} - \lambda\gamma^* \mathcal{K} A' A \mathcal{K}' - \lambda\left(1-\alpha^*\right) \mathcal{K}\left[\left(\kappa_1^* A' - B'h\right)\left(\kappa_1^* A - h'B\right) - \left(A' - B'h\right)\left(A - h'B\right)\right] \mathcal{K}'\right)^{-1},$$

with the scalar $\hat{w}$ given by

$$\hat{w} \equiv \frac{\tau_\mu}{1 - \lambda(1-\alpha^*)\left(A - h'B\right) \mathcal{K}' \tilde{\Omega}_\mu \mathcal{K}\left(A' - B'h\right)}.$$

Then, the equilibrium h satisfies

$$h' = A \Delta B' \left(B \Delta B'\right)^{-1}.$$

Proof. It follows from Lemma B.6 that

$$\left(A - h'B\right) \Omega_\Gamma B' = 0,$$

and from the definition of Ω_Γ and $\tilde{\Omega}_\mu$ we have that

$$\Omega_\Gamma = \Gamma\Omega + \mathcal{K}' \left(\tilde{\Omega}_\mu^{-1} - \lambda(1-\alpha^*) \mathcal{K}\left(A' - B'h\right)\left(A - h'B\right) \mathcal{K}'\right)^{-1} \mathcal{K}.$$

It is then sufficient to show that

$$(A-h'B)\left(\Gamma\Omega+\hat{w}\tau_\mu^{-1}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\right)=(A-h'B)\left(\Gamma\Omega+\mathcal{K}'\left(\tilde{\Omega}_\mu^{-1}-\lambda\left(1-\alpha^*\right)\mathcal{K}\left(A'-B'h\right)\left(A-h'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}\right),$$

or, equivalently,

$$\begin{aligned}\hat{w}\tau_\mu^{-1}\left(A-h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}&=\left(A-h'B\right)\mathcal{K}'\left(\tilde{\Omega}_\mu^{-1}-\lambda\left(1-\alpha^*\right)\mathcal{K}\left(A'-B'h\right)\left(A-h'B\right)\mathcal{K}'\right)^{-1}\mathcal{K}\\&=\left(A-h'B\right)\mathcal{K}'\left(\mathrm{I}_u-\lambda\left(1-\alpha^*\right)\tilde{\Omega}_\mu\mathcal{K}\left(A'-B'h\right)\left(A-h'B\right)\mathcal{K}'\right)^{-1}\tilde{\Omega}_\mu\mathcal{K}.\end{aligned}$$

Thus, it is sufficient to establish that

$$\hat{w}\tau_\mu^{-1}\left(A-h'B\right)\mathcal{K}'=\left(A-h'B\right)\mathcal{K}'\left(\mathrm{I}_u-\lambda\left(1-\alpha^*\right)\tilde{\Omega}_\mu\mathcal{K}\left(A'-B'h\right)\left(A-h'B\right)\mathcal{K}'\right)^{-1}.$$

It follows that

$$\hat{w}\tau_\mu^{-1}\left(A-h'B\right)\mathcal{K}'\left(\mathrm{I}_u-\lambda\left(1-\alpha^*\right)\tilde{\Omega}_\mu\mathcal{K}\left(A'-B'h\right)\left(A-h'B\right)\mathcal{K}'\right)=\left(A-h'B\right)\mathcal{K}',$$

which can be rewritten as

$$\hat{w}\tau_\mu^{-1}\left(1-\lambda\left(1-\alpha^*\right)\left(A-h'B\right)\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K}\left(A'-B'h\right)\right)\left(A-h'B\right)\mathcal{K}'=\left(A-h'B\right)\mathcal{K}'.$$

The definition of $\hat{w}$ then yields the result. □

Lemma B.8. *Let*

$$\omega\equiv-\frac{\gamma}{\left(1-\alpha^*\right)\left(1-\kappa_1^*\right)},\quad v_1\equiv-\frac{\gamma\lambda}{\omega}\mathcal{K}A',\quad v_2\equiv\mathcal{K}\left(\omega A'-B'h\right),$$

and

$$c_{ij}\equiv v_i'\Omega_\mu v_j,\quad \textit{for } i,j\in\{1,2\},\quad \textit{and}\quad s_i\equiv\left(A-h'B\right)\mathcal{K}'\Omega_\mu v_i,\quad \textit{for } i\in\{1,2\}.$$

Further, define

$$\tilde{\Delta}\equiv\Gamma\Omega+\tilde{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu\mathcal{K},$$

with the scalar $\tilde{w}$ given by

$$\tilde{w}=\left(1+\frac{c_{11}s_2-\left(1+c_{12}\right)s_1}{\left(1+c_{12}\right)\left(1+c_{21}\right)-c_{11}c_{22}}\right)\hat{w},$$

and let the scalar $\tilde{r}$ be given by

$$\tilde{r}=-\frac{\frac{\lambda\gamma}{\omega}\left(c_{22}s_1-\left(1+c_{21}\right)s_2\right)+\left(1-\omega\right)\left(c_{11}s_2-\left(1+c_{12}\right)s_1\right)}{\left(1+c_{12}\right)\left(1+c_{21}\right)-c_{11}c_{22}+c_{11}s_2-\left(1+c_{12}\right)s_1}\frac{\tilde{w}}{1+\tilde{w}}.$$

Then, the equilibrium h satisfies

$$h'=\left(1+\tilde{r}\right)A\tilde{\Delta}B'\left(B\tilde{\Delta}B'\right)^{-1}.$$

Proof. From the definition of $\tilde{\Omega}_\mu$ in Lemma B.7, we have that

$$\tilde{\Omega}_\mu = \left(\Omega_\mu^{-1} + \lambda\left(1-\alpha^*\right)\left(1-\kappa_1^*\right)\mathcal{K}\left[A'\left(\omega A - h'B\right) + \left(\omega A' - B'h\right)A\right]\mathcal{K}'\right)^{-1},$$

with

$$\omega \equiv \frac{\left(1-\alpha^*\right)\left(1-(\kappa_1^*)^2\right)-\gamma^*}{\left(1-\alpha^*\right)\left(1-\kappa_1^*\right)} = -\frac{\gamma}{\left(1-\alpha^*\right)\left(1-\kappa_1^*\right)}.$$

Thus, defining

$$v_1 \equiv -\frac{\gamma\lambda}{\omega}\mathcal{K}A', \quad \text{and} \quad v_2 \equiv \mathcal{K}\left(\omega A' - B'h\right),$$

we can write

$$\tilde{\Omega}_\mu = (\Omega_\mu^{-1} + v_1 v_2' + v_2 v_1')^{-1},$$

and applying the Sherman-Morrison formula twice, we obtain

$$\tilde{\Omega}_\mu = \Omega_\mu + \frac{c_{11}\Omega_\mu v_2 v_2'\Omega_\mu + c_{22}\Omega_\mu v_1 v_1'\Omega_\mu - \left(1+c_{12}\right)\Omega_\mu v_1 v_2'\Omega_\mu - \left(1+c_{21}\right)\Omega_\mu v_2 v_1'\Omega_\mu}{\left(1+c_{12}\right)\left(1+c_{21}\right) - c_{11}c_{22}},$$

with

$$c_{ij} \equiv v_i'\Omega_\mu v_j, \quad \text{for } i,j \in \{1,2\}.$$

Thus, from the definition of Δ in Lemma B.7 and defining

$$\bar{\Delta} \equiv \Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu\mathcal{K},$$

we have that

$$\Delta = \Gamma\Omega + \hat{w}\tau_\mu^{-1}\mathcal{K}'\tilde{\Omega}_\mu\mathcal{K} = \bar{\Delta} + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K},$$

with

$$V \equiv \frac{c_{11}v_2 v_2' + c_{22}v_1 v_1' - \left(1+c_{12}\right)v_1 v_2' - \left(1+c_{21}\right)v_2 v_1'}{\left(1+c_{12}\right)\left(1+c_{21}\right) - c_{11}c_{22}}.$$

Hence, it follows from the result in Lemma B.7 that

$$h' = A\left(\bar{\Delta} + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}\right)B'\left[B\left(\bar{\Delta} + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}\right)B'\right]^{-1},$$

and, therefore,

$$h'\left[B\left(\bar{\Delta} + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}\right)B'\right] = A\left(\bar{\Delta} + \hat{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}\right)B'.$$

Rearranging, we get

$$h'B\bar{\Delta}B' + \hat{w}\tau_\mu^{-1}h'B\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}B' = A\bar{\Delta}B' + \hat{w}\tau_\mu^{-1}A\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}B',$$

and right-multiplying both side by $\left(B\bar{\Delta}B'\right)^{-1}$ yields

$$h' = A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \hat{w}\tau_\mu^{-1}\left(A - h'B\right)\mathcal{K}'\Omega_\mu V\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1}.$$

Defining

$$s_i \equiv (A - h'B)\,\mathcal{K}'\Omega_\mu v_i, \quad \text{for } i \in \{1,2\},$$

we obtain

$$\begin{aligned} h' &= A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \hat{w}\tau_\mu^{-1}\frac{\left(c_{22}s_1 - (1+c_{21})\,s_2\right)v_1' + \left(c_{11}s_2 - (1+c_{12})\,s_1\right)v_2'}{(1+c_{12})\,(1+c_{21}) - c_{11}c_{22}}\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1} \\ &= A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \hat{w}\tau_\mu^{-1}\alpha_1 A\mathcal{K}'\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1} + \hat{w}\tau_\mu^{-1}\alpha_2 h'B\mathcal{K}'\Omega_\mu\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1} \end{aligned}$$

with

$$\alpha_1 \equiv \frac{-\left(c_{22}s_1 - (1+c_{21})\,s_2\right)\frac{\gamma\lambda}{\omega} + \left(c_{11}s_2 - (1+c_{12})\,s_1\right)\omega}{(1+c_{12})\,(1+c_{21}) - c_{11}c_{22}}, \quad \text{and} \quad \alpha_2 \equiv -\frac{\left(c_{11}s_2 - (1+c_{12})\,s_1\right)}{(1+c_{12})\,(1+c_{21}) - c_{11}c_{22}}.$$

Next, notice that

$$\mathcal{K}'\Omega_\mu\mathcal{K} = \tau_\mu\mathcal{K}'\mathcal{K}\Omega,$$

and

$$\bar{\Delta} = \Big((1+\hat{w})\,\mathcal{K}'\mathcal{K} + (1-\alpha)^{-1}\,(\mathrm{I}_m - \mathcal{K}'\mathcal{K})\Big)\,\Omega,$$

so we have

$$\mathcal{K}'\Omega_\mu\mathcal{K} = \frac{\tau_\mu}{1+\hat{w}}\mathcal{K}'\mathcal{K}\bar{\Delta}.$$

Thus, it follows that

$$h' = \left(1 + \alpha_1\frac{\hat{w}}{1+\hat{w}}\right)A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \alpha_2\frac{\hat{w}}{1+\hat{w}}h'B\mathcal{K}'\mathcal{K}\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1},$$

or

$$h' = \beta_1 A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \beta_2 h'B\mathcal{K}'\mathcal{K}\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1}$$

with

$$\beta_1 \equiv 1 + \alpha_1\frac{\hat{w}}{1+\hat{w}}, \quad \text{and} \quad \beta_2 = \alpha_2\frac{\hat{w}}{1+\hat{w}}$$

Define

$$\tilde{\Delta} \equiv (\mathrm{I}_m - \beta_2\mathcal{K}'\mathcal{K})\,\bar{\Delta},$$

and guess that

$$h' = \frac{\beta_1}{1-\beta_2}A\tilde{\Delta}B'\left(B\tilde{\Delta}B'\right)^{-1}.$$

It follows that

$$\begin{aligned} \frac{\beta_1}{1-\beta_2}A\tilde{\Delta}B'\left(B\tilde{\Delta}B'\right)^{-1}B\bar{\Delta}B' &= \beta_1 A\bar{\Delta}B' + \frac{\beta_1}{1-\beta_2}\beta_2 A\tilde{\Delta}B'\left(B\tilde{\Delta}B'\right)^{-1}B\mathcal{K}'\mathcal{K}\bar{\Delta}B' \\ \frac{\beta_1}{1-\beta_2}A\tilde{\Delta}B'\left(B\tilde{\Delta}B'\right)^{-1}B\,(\mathrm{I}_m - \beta_2\mathcal{K}'\mathcal{K})\,\bar{\Delta}B' &= \beta_1 A\bar{\Delta}B' \\ \beta_1 A\tilde{\Delta}B' &= \beta_1\,(1-\beta_2)\,A\bar{\Delta}B', \end{aligned}$$

which verifies the guess, since $A\tilde{\Delta} = (1-\beta_2)\, A\bar{\Delta}$. Finally, we have that

$$\begin{aligned}\tilde{\Delta} &= \left(\mathrm{I}_m - \beta_2\mathcal{K}'\mathcal{K}\right)\Big((1+\hat{w})\,\mathcal{K}'\mathcal{K} + (1-\alpha)^{-1}\left(\mathrm{I}_m - \mathcal{K}'\mathcal{K}\right)\Big)\,\Omega \\ &= \Big((1-\beta_2)\,(1+\hat{w})\,\mathcal{K}'\mathcal{K} + (1-\alpha)^{-1}\left(\mathrm{I}_m - \mathcal{K}'\mathcal{K}\right)\Big)\,\Omega \\ &= \Gamma\Omega + \tilde{w}\tau_\mu^{-1}\mathcal{K}'\Omega_\mu\mathcal{K},\end{aligned}$$

with

$$\tilde{w} = (1-\beta_2)\,(1+\hat{w}) - 1 = \left(1 - \alpha_2\frac{\hat{w}}{1+\hat{w}}\right)(1+\hat{w}) - 1 = (1-\alpha_2)\hat{w}.$$

and

$$\tilde{r} = \frac{\beta_1}{1-\beta_2} - 1 = \frac{1+\alpha_1\frac{\hat{w}}{1+\hat{w}}}{1-\alpha_2\frac{\hat{w}}{1+\hat{w}}} - 1 = \frac{\alpha_1+\alpha_2}{1-\alpha_2}\frac{\tilde{w}}{1+\tilde{w}}.$$

Substituting the definitions of α_1 and α_2 yields the result. □

Parts 1 and 2 of Proposition B.3. Given the result in Lemma B.8, we are left with taking the limit as $T \to \infty$ of the truncated problem. In particular, we have that

$$\begin{aligned}\lim_{T\to\infty} c_{11} &= \tau_\mu\left(\frac{\gamma\lambda}{\omega}\right)^2\mathbb{V}(\xi_t)\,, \quad \lim_{T\to\infty} c_{12} = \lim_{T\to\infty} c_{21} = -\tau_\mu\frac{\gamma\lambda}{\omega}\mathbb{COV}(\omega\xi_t - K_t, \xi_t)\,, \\ \lim_{T\to\infty} c_{22} &= \tau_\mu\mathbb{V}(\omega\xi_t - K_t)\,, \quad \lim_{T\to\infty} s_1 = -\tau_\mu\frac{\gamma\lambda}{\omega}\mathbb{COV}(\xi_t - K_t, \xi_t)\,, \quad \text{and} \\ \lim_{T\to\infty} s_2 &= \tau_\mu\mathbb{COV}(\omega\xi_t - K_t, \xi_t - K_t)\,.\end{aligned}$$

Notice that

$$\hat{w} = \frac{\tau_\mu}{1-\lambda(1-\alpha^*)\left\{(A-h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}\,(A'-B'h) + \frac{c_{11}s_2^2 + c_{22}s_1^2 - (2+c_{12}+c_{21})s_1s_2}{(1+c_{12})(1+c_{21}) - c_{11}c_{22}}\right\}}.$$

Let $w \equiv \lim_{T\to\infty}\tilde{w}$, and $r \equiv \lim_{T\to\infty}\tilde{r}$. Using equations $\omega = \frac{\gamma}{(1-\alpha^*)\left(1-\kappa_1^*\right)}$, $\alpha^* = \alpha - \psi$, and $(1-\alpha^*)\left(1-\kappa_1^*\right) = \phi$, in order to return to primitive parameters, it follows that

$$w = \frac{\tau_\mu\left(1+\lambda\left(1-\alpha+\psi\right)r\mathbb{V}(\xi_t)\left(1-\mathcal{S}\right)\right)}{1-\lambda\left(1-\alpha+\psi\right)\tau_\mu\left(\mathbb{V}(\xi_t - K_t) + r\mathbb{V}(\xi_t)\left(1-\mathcal{S}\right)\right) + \nu_1},$$

and

$$r = \frac{\gamma\lambda\tau_\mu\mathbb{V}(\xi_t)\left(1+\nu_2\right)}{1-\gamma\lambda\tau_\mu\mathbb{V}(\xi_t)\left(1+\nu_3\right)}\frac{w}{1+w}\left(1-\mathcal{S}\right),$$

with

$$
\begin{aligned}
\nu_1 &\equiv \frac{\lambda^2\phi^2\tau_\mu^2\left(\mathbb{V}\left(\xi_t\right)^2\left(1-\mathcal{S}\right)^2-\mathbb{V}\left(\xi_t\right)\mathbb{V}\left(\xi_t-K_t\right)\right)-\lambda\phi\tau_\mu\mathbb{V}\left(\xi_t\right)\left(1-\mathcal{S}\right)}{1-\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)\left(2\gamma-\phi\left(1+\mathcal{S}\right)\right)},\\
\nu_2 &\equiv 1-\frac{\phi}{\gamma}\left(2-\frac{\mathbb{V}\left(\xi_t-K_t\right)}{\mathbb{V}\left(\xi_t\right)\left(1-\mathcal{S}\right)}\right),\\
\nu_3 &\equiv 1-\frac{\phi}{\gamma}\left(1+\mathcal{S}\right).
\end{aligned}
$$

This completes the proof of parts 1 and 2 of Proposition B.3.

Part 3 of Proposition B.3. Next, we switch focus to the level of the $\mathcal{B} \equiv \lim_{T\to\infty} h_0$. From equation (A.5) and the definition of π, we have that

$$
\begin{aligned}
\left(1-\alpha\right)h_0 = {}& \left[\left(1-\alpha\right)\left(A-\mathrm{T}B\right)+\alpha h'\left(B\Lambda-\mathrm{H}B\right)\right]\\
&\times\mathcal{K}'S\left[-\lambda\left(1-\alpha^*\right)h_0\mathcal{K}\left(A'-B'h\right)+\lambda\chi\mathcal{K}A'+\lambda\varphi\mathcal{K}B'h\right],
\end{aligned}
$$

which, using $\varphi=(1-\alpha^*)\kappa_0^*$ and defining $\chi^*\equiv\chi+(1-\alpha^*)\kappa_0^*$, can be rewritten as

$$
\begin{aligned}
\left(1-\alpha\right)h_0 = {}& \left[\left(1-\alpha\right)\left(A-\mathrm{T}B\right)+\alpha h'\left(B\Lambda-\mathrm{H}B\right)\right]\\
&\times\mathcal{K}'S\left[-\lambda\left(1-\alpha^*\right)\left(h_0+\kappa_0^*\right)\mathcal{K}\left(A'-B'h\right)+\lambda\chi^*\mathcal{K}A'\right].
\end{aligned}
\tag{B.10}
$$

It is straightforward to see there exists a unique h_0 that satisfies this equation. We postulate that there exists $\tilde{\mu}$ such that

$$
\left(1-\alpha\right)h_0=\left[(1-\alpha)A+\alpha h'B\Lambda-h'B\right]\mathcal{K}'\tilde{\mu},
$$

so that solving for $\tilde{\mu}$ pins down the unique h_0. To proceed, first replace the guess for h_0 on the RHS of equation (B.10),

$$
\begin{aligned}
\mathrm{RHS} \equiv {}& \left[\left(1-\alpha\right)\left(A-\mathrm{T}B\right)+\alpha h'\left(B\Lambda-\mathrm{H}B\right)\right]\mathcal{K}'S\left[-\lambda\left(1-\alpha^*\right)\left(h_0+\kappa_0^*\right)\mathcal{K}\left(A'-B'h\right)+\lambda\chi^*\mathcal{K}A'\right]\\
= {}& \left[\left(1-\alpha\right)\left(A-\mathrm{T}B\right)+\alpha h'\left(B\Lambda-\mathrm{H}B\right)\right]\mathcal{K}'S\\
&\times\left\{-\lambda\mathcal{K}\left(A'-B'h\right)\left(1-\alpha^*\right)\left\{\frac{\left[\left(1-\alpha\right)\left(A-h'B\right)+\alpha h'B\left(\Lambda-\mathrm{I}_m\right)\right]\mathcal{K}'\tilde{\mu}}{1-\alpha}+\kappa_0^*\right\}+\lambda\chi^*\mathcal{K}A'\right\}
\end{aligned}
$$

Next, for the LHS of the equation, we have that

$$
\mathrm{LHS}\equiv\left(1-\alpha\right)h_0=\left[(1-\alpha)A+\alpha h'B\Lambda-h'B\right]\mathcal{K}'\tilde{\mu},
$$

and, substituting the last h using equation (A.4), it follows that

$$\begin{aligned}
\text{LHS} = & \left[(1-\alpha)(A-\text{T}B)+\alpha h'(B\Lambda-\text{H}B)\right]\left[\text{I}_\text{m}-\mathcal{K}'\text{S}\mathcal{K}\text{B}'\left(\text{B}\Omega\text{B}'\right)^{-1}\text{B}\right]\mathcal{K}'\tilde{\mu}\\
= & \left[(1-\alpha)(A-\text{T}B)+\alpha h'(B\Lambda-\text{H}B)\right]\mathcal{K}'S\left[S^{-1}-\mathcal{K}B'\left(B\Omega B'\right)^{-1}B\mathcal{K}'\right]\tilde{\mu}\\
= & \left[(1-\alpha)(A-\text{T}B)+\alpha h'(B\Lambda-\text{H}B)\right]\mathcal{K}'S\\
& \times\left\{\Omega_\mu^{-1}-\lambda\left[(1-\alpha^*)\mathcal{K}(\kappa_1^*A'-B'h)(\kappa_1^*A-h'B)\mathcal{K}'+\gamma^*\mathcal{K}A'A\mathcal{K}'\right]\right\}\tilde{\mu},
\end{aligned}$$

where the last equality uses the definition of S. Putting these results together, we have that

$$\begin{aligned}
\text{LHS}-\text{RHS} = & \left[(1-\alpha)(A-\text{T}B)+\alpha h'(B\Lambda-\text{H}B)\right]\mathcal{K}'S\\
& \times\Big\{\left\{\Omega_\mu^{-1}-\lambda\left[(1-\alpha^*)\mathcal{K}(\kappa_1^*A'-B'h)(\kappa_1^*A-h'B)\mathcal{K}'+\gamma^*\mathcal{K}A'A\mathcal{K}'\right]\right\}\tilde{\mu}\\
& +\lambda(1-\alpha^*)\mathcal{K}(A'-B'h)(A-h'B)\mathcal{K}'\tilde{\mu}+\lambda(1-\alpha^*)\kappa_0^*\mathcal{K}(A'-B'h)-\lambda\chi^*\mathcal{K}A'\Big\},
\end{aligned}$$

where we used the fact that $\mathcal{K}(A'-B'h)h'B(\Lambda-\text{I}_\text{m})\mathcal{K}'=0$. Thus, a sufficient condition for $\text{LHS}-\text{RHS}=0$ is

$$\begin{aligned}
& \left\{\Omega_\mu^{-1}-\lambda\left[(1-\alpha^*)\mathcal{K}(\kappa_1^*A'-B'h)(\kappa_1^*A-h'B)\mathcal{K}'+\gamma^*\mathcal{K}A'A\mathcal{K}'\right]\right\}\tilde{\mu}\\
& +\lambda(1-\alpha^*)\mathcal{K}(A'-B'h)(A-h'B)\mathcal{K}'\tilde{\mu}+\lambda(1-\alpha^*)\kappa_0^*\mathcal{K}(A'-B'h)-\lambda\chi^*\mathcal{K}A' = 0.
\end{aligned}$$

Notice that, using the definitions from Lemma B.8, this equation can be rewritten as

$$\left\{\Omega_\mu^{-1}+v_1v_1'+v_2v_2'\right\}\tilde{\mu}=-\lambda(1-\alpha^*)\kappa_0^*\mathcal{K}(A'-B'h)+\lambda\chi^*\mathcal{K}A'.$$

It follows that

$$\tilde{\mu}=\left\{\Omega_\mu+\Omega_\mu V\Omega_\mu\right\}\left\{-\lambda(1-\alpha^*)\kappa_0^*\mathcal{K}(A'-B'h)+\lambda\chi^*\mathcal{K}A'\right\},$$

and, therefore,

$$\begin{aligned}
h_0 &= (1-\alpha)^{-1}\left[(1-\alpha)A+\alpha h'B\Lambda-h'B\right]\mathcal{K}'\tilde{\mu}\\
&= (A-h'B)\mathcal{K}'\tilde{\mu}\\
&= (A-h'B)\mathcal{K}'\left\{\Omega_\mu+\Omega_\mu V\Omega_\mu\right\}\left\{-\lambda(1-\alpha^*)\kappa_0^*\mathcal{K}(A'-B'h)+\lambda\chi^*\mathcal{K}A'\right\}\\
&= -\lambda(1-\alpha^*)\kappa_0^*\left\{(A-h'B)\mathcal{K}'\Omega_\mu\mathcal{K}(A'-B'h)+\frac{c_{11}s_2^2+c_{22}s_1^2-(2+c_{12}+c_{21})s_1s_2}{(1+c_{12})(1+c_{21})-c_{11}c_{22}}\right\}\\
&+\lambda\chi^*\left\{(A-h'B)\mathcal{K}'\Omega_\mu\mathcal{K}A'+\frac{c_{11}s_2z_2+c_{22}s_1z_1-(1+c_{12})s_1z_2-(1+c_{21})s_2z_1}{(1+c_{12})(1+c_{21})-c_{11}c_{22}}\right\},
\end{aligned}$$

with

$$z_i\equiv A\mathcal{K}'\Omega_\mu v_i,\quad \text{for } i\in\{1,2\}.$$

Notice that we have the following limits

$$\lim_{T\to\infty}z_1=-\tau_\mu\frac{\gamma\lambda}{\omega}\mathbb{V}(\xi_t),\quad\text{and}\quad\lim_{T\to\infty}z_2=\tau_\mu\mathbb{COV}(\omega\xi_t-K_t,\xi_t).$$

Therefore, using $\chi^* = \chi + \varphi$, and $(1-\alpha^*)\,\kappa_0^* = \varphi$, we obtain the bias as a function of primitive parameters,

$$\mathcal{B} = \frac{\chi\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)\left(1-\mathcal{S}\right)+\nu_4}{1-\gamma\lambda\tau_\mu\mathbb{V}\left(\xi_t\right)+\nu_5},$$

with

$$\begin{aligned}
\nu_4 \equiv\ & \lambda\varphi\tau_\mu\left(\mathbb{V}\left(\xi_t\right)\left(1-\mathcal{S}\right)-\mathbb{V}\left(\xi_t-K_t\right)\right) \\
& -\lambda^2\tau_\mu^2\left(\phi\left(\chi-\varphi\right)+2\gamma\varphi\right)\left(\mathbb{V}\left(\xi_t\right)^2\left(1-\mathcal{S}\right)^2-\mathbb{V}\left(\xi_t\right)\mathbb{V}\left(\xi_t-K_t\right)\right) \\
\nu_5 \equiv\ & \lambda\tau_\mu\mathbb{V}\left(\xi_t\right)\left(2\phi\mathcal{S}-\gamma\right)+\lambda^2\tau_\mu^2\phi^2\left(\mathbb{V}\left(\xi_t\right)^2\left(1-\mathcal{S}\right)^2-\mathbb{V}\left(\xi_t\right)\mathbb{V}\left(\xi_t-K_t\right)\right),
\end{aligned}$$

which completes the proof of part 3 of the proposition. □

B.3 Multiple aggregate shocks

Consider the same setup described in Section 3, but suppose that the common fundamental, ξ_t, is now driven by a $Z\times 1$ vector of shocks, η_t according to the following stochastic process:

$$\xi_t = a(L)\eta_t, \quad \text{with} \quad \eta_t \sim \mathcal{N}(0,\Sigma_\eta),$$

where $a(L)$ is a polynomial in the lag operator L. In the objective environment, η_t is normally distributed with mean zero: $\mu_t = 0$. Subjectively, agents believe that η_t is drawn from a Gaussian distribution with the same covariance matrix, Σ_η, but there is uncertainty about its prior mean, denoted by the $Z\times 1$ vector μ_t. Ambiguity about ξ_t is then captured by the perception that

$$\eta_t \sim \mathcal{N}(\mu_t,\Sigma_\eta), \quad \text{and} \quad \mu_t \sim \mathcal{N}(0,\Sigma_\mu).$$

In Section 3, the degree of ambiguity is captured by the σ_μ^2. Here, the covariance matrix Σ_μ plays this role. Without loss of generality, we assume that Σ_η and Σ_μ are diagonal matrices, that is $\Sigma_\eta = \text{diag}(\sigma^2_{\eta,1},\ldots,\sigma^2_{\eta,Z})$ and $\Sigma_\mu = \text{diag}(\sigma^2_{\mu,1},\ldots,\sigma^2_{\mu,Z})$.

Auxiliary forecasting problem Consider the following pure forecasting problem, which we later link back to the economy with ambiguity.

Definition B.3. *The $(w,\alpha,\{r_i\}_{i=1}^Z)$-modified signal process is given by*

$$\begin{aligned}
\tilde{\xi}_t &= a(L)\,\text{diag}(1+r_1,\ldots,1+r_q)\tilde{\eta}_t, && \textit{with} \quad \tilde{\eta}_t \sim \mathcal{N}(0,\Sigma_\eta + w\Sigma_\mu), \\
\tilde{x}_{it} &= m(L)\tilde{\eta}_t + n(L)\tilde{\epsilon}_{it}, && \textit{with} \quad \tilde{\epsilon}_{it} \sim \mathcal{N}(0,(1-\alpha)^{-1}\Sigma),
\end{aligned}$$

where w is a non-negative scalar and α is the degree of complementarity. Let the optimal Bayesian forecast be given by

$$\tilde{\mathbb{E}}_{it}[\tilde{\xi}_t] = p(L;w,\alpha,\{r_i\}_{i=1}^Z)\tilde{x}_{it}.$$

This modified signal process is analogous to the baseline. The adjustment w to the volatility of η_t is the counterpart to $\tilde{w} = w\tau_\mu^{-1}$ in the univariate baseline, that is $\Sigma_\eta + w\Sigma_\mu$ is the counterpart of $(1+w)\sigma_\eta^2 = (1+\tilde{w}\tau_\mu)\sigma_\eta^2 = \sigma_\eta^2 + \tilde{w}\sigma_\mu^2$. Further, the amplification factor, $(1+r)$ in the univariate case, has now been incorporated into this modified signal process since in the multivariate case each shock requires a potentially different adjustment, before being put together into a modified fundamental. So, $p(L; w, \alpha, \{r_i\}_{i=1}^Z)$ here is the counterpart of $(1+r)p(L; w, \alpha)$ in the univariate case. To proceed we need the additional following definitions.

Definition B.4. *Define the μ-modified fundamental and (unbiased) aggregate action as*

$$\xi_t^\mu = a(L)\mu_t, \quad \textit{and} \quad K_t^\mu = p(L; w, \alpha, \{r_i\}_{i=1}^Z)\mu_t,$$

and the μ-modified aggregate sensitivity to signals as

$$\mathcal{S}^\mu \equiv 1 - \frac{\mathbb{COV}\left(\xi_t^\mu - K_t^\mu, \xi_t^\mu\right)}{\mathbb{V}\left(\xi_t^\mu\right)}.$$

We can then prove the following proposition.

Proposition B.4. *The linear strategy in equilibrium takes the following form*

$$g(x_i^t) = p(L; w, \alpha, \{r_i\}_{i=1}^Z)x_{it} + \mathcal{B}.$$

1. *The polynomial matrix $p(L; w, \alpha, \{r_i\}_{i=1}^Z)$ is the Bayesian forecasting rule with the $(w, \alpha, \{r_i\}_{i=1}^Z)$-modified signal process and w satisfies*

$$w = \frac{1}{1 - \lambda(1-\alpha)\left(\mathbb{V}\left(\xi_t^\mu - K_t^\mu\right) + \frac{\lambda\gamma\mathbb{V}(\xi_t^\mu)^2(1-\mathcal{S}^\mu)^2}{1-\lambda\gamma\mathbb{V}(\xi_t^\mu)}\right)};$$

2. *For all $i \in \{1, \dots, Z\}$, the additional amplification, r_i, satisfies*

$$r_i = \gamma\frac{\lambda\mathbb{V}\left(\xi_t^\mu\right)}{1-\lambda\gamma\mathbb{V}\left(\xi_t^\mu\right)}\frac{w\tau_{\mu,i}}{1+w\tau_{\mu,i}}(1-\mathcal{S}^\mu);$$

3. *The level of bias, $\mathcal{B}$, satisfies*

$$\mathcal{B} = \chi\frac{\lambda\mathbb{V}\left(\xi_t^\mu\right)}{1-\lambda\gamma\mathbb{V}\left(\xi_t^\mu\right)}\left(1-\mathcal{S}^\mu\right).$$

Proof of Proposition B.4. The truncated version of the problem is analogous to the case with one common shock, with the following adjustments: (1) the size of the vector of aggregate common shocks must be set to $u \equiv Z(T+1)$; (2) the size of the vector of all shocks becomes $m \equiv (Z+N)(T+1)$; (3) instead of $\Omega_\eta = \mathrm{I}_u\,\sigma_\eta^2$ and $\Omega_\mu = \mathrm{I}_u\,\sigma_\mu^2$, we now have $\Omega_\eta = \mathrm{I}_{T+1} \otimes \Sigma_\eta$ and $\Omega_\mu = \mathrm{I}_{T+1} \otimes \Sigma_\mu$. These modifications do not affect in any way the results in Lemmas A.1, A.2, and A.3. However, Lemma A.4 relies on the fact that $\Omega_\eta = \mathrm{I}_u\,\sigma_\eta^2$ and $\Omega_\mu = \mathrm{I}_u\,\sigma_\mu^2$. The following lemma provides the relevant analogous result.

Lemma B.9. *Define*

$$\bar{\Delta} \equiv \Gamma\Omega + \hat{w}\mathcal{K}'\Omega_\mu\mathcal{K},$$

with the scalar $\hat{w}$ *given by*

$$\hat{w}=\frac{1}{1-\lambda\left(1-\alpha\right)\left(A-h'B\right)\mathcal{K}'\left(\Omega_{\mu}+\frac{\lambda\gamma\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}}{1-\lambda\gamma A\mathcal{K}'\Omega_{\mu}\mathcal{K}A'}\right)\mathcal{K}\left(A'-B'h\right)}.$$

Also, let the diagonal matrix $\hat{R}$ *be given by*

$$\hat{R}\equiv \mathrm{I}_{T+1}\otimes \mathrm{diag}\left(\hat{r}_1,\ldots,\hat{r}_Z\right),$$

with the scalars $\hat{r}_i$*, for* $i\in\{1,\ldots,Z\}$*, given by*

$$\hat{r}_i\equiv\frac{\hat{w}\tau_{\mu,i}}{1+\hat{w}\tau_{\mu,i}}\left(\frac{\lambda\gamma}{1-\lambda\gamma A\mathcal{K}'\Omega_{\mu}\mathcal{K}A'}\right)\left(A-h'B\right)\mathcal{K}'\Omega_{\mu}\mathcal{K}A',\quad \textit{with}\quad \tau_{\mu,i}\equiv\frac{\sigma^2_{\mu,i}}{\sigma^2_{\eta,i}}.$$

Then, the equilibrium h *satisfies*

$$h'=A\left(\mathrm{I}_m+\hat{R}\right)\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1}.$$

Proof. From the definition of $\tilde{\Omega}_{\mu}$ and Δ in Lemma A.3, we have that

$$\tilde{\Omega}_{\mu}\equiv\left(\Omega_{\mu}^{-1}-\lambda\gamma\mathcal{K}A'A\mathcal{K}'\right)^{-1}=\Omega_{\mu}+\frac{\lambda\gamma\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}}{1-\lambda\gamma A\mathcal{K}'\Omega_{\mu}\mathcal{K}A'}.$$

Thus, it follows that

$$\Delta\equiv\Gamma\Omega+\hat{w}\mathcal{K}'\tilde{\Omega}_{\mu}\mathcal{K}=\bar{\Delta}+\hat{w}\mathcal{K}'\left(\frac{\lambda\gamma\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}}{1-\lambda\gamma A\mathcal{K}'\Omega_{\mu}\mathcal{K}A'}\right)\mathcal{K}=\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}\right)\mathcal{K},$$

with $s\equiv\lambda\gamma/(1-\lambda\gamma A\mathcal{K}'\Omega_{\mu}\mathcal{K}A')$. Hence, it follows from the result in Lemma A.3 that

$$h'=A\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}\right)\mathcal{K}\right)B'\left[B\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}\right)\mathcal{K}\right)B'\right]^{-1},$$

and, therefore,

$$h'\left[B\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}\right)\mathcal{K}\right)B'\right]=A\left(\bar{\Delta}+s\hat{w}\mathcal{K}'\left(\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}\right)\mathcal{K}\right)B'.$$

Rearranging, we get

$$h'B\bar{\Delta}B'+s\hat{w}h'B\mathcal{K}'\left(\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}\right)\mathcal{K}B'=A\bar{\Delta}B'+s\hat{w}A\mathcal{K}'\left(\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}\right)\mathcal{K}B',$$

and right-multiplying both sides by $\left(B\bar{\Delta}B'\right)^{-1}$ yields

$$\begin{aligned}h'&=A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1}+s\hat{w}\left(A-h'B\right)\mathcal{K}'\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1}\\&=A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1}+\hat{z}\hat{w}A\mathcal{K}'\Omega_{\mu}\mathcal{K}B'\left(B\bar{\Delta}B'\right)^{-1}.\end{aligned}$$

Next, notice that

$$\mathcal{K}'\Omega_{\mu}\mathcal{K}=\mathcal{K}'\Omega_{\mu}\Omega_{\eta}^{-1}\mathcal{K}\Omega,$$

and

$$\bar{\Delta} = \Big(\mathcal{K}'(\mathrm{I}_u + \hat{w}\Omega_\mu\Omega_\eta^{-1})\mathcal{K} + (1-\alpha)^{-1}\left(\mathrm{I}_m - \mathcal{K}'\mathcal{K}\right)\Big)\,\Omega,$$

so we have

$$\mathcal{K}'\Omega_\mu\mathcal{K} = \mathcal{K}'\left(\Omega_\mu(\Omega_\eta + \hat{w}\Omega_\mu)^{-1}\right)\mathcal{K}\bar{\Delta}.$$

Thus, it follows that

$$h' = A\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} + \hat{z}\hat{w}A\mathcal{K}'\left(\Omega_\mu(\Omega_\eta + \hat{w}\Omega_\mu)^{-1}\right)\mathcal{K}\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1},$$

with the scalar $\hat{z}$ given by

$$\hat{z} \equiv \left(\frac{\lambda\gamma}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\right)(A - h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A'.$$

Further, we can write

$$h' = A\,(\mathrm{I}_m + \hat{R})\,\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1},$$

with

$$\begin{aligned}
\hat{R} &= \mathcal{K}'\left(\hat{z}\hat{w}\Omega_\mu(\Omega_\eta + \hat{w}\Omega_\mu)^{-1}\right)\mathcal{K}\\
&= \mathcal{K}'\left(\hat{z}\hat{w}(\mathrm{I}_{T+1}\otimes\Sigma_\mu)((\mathrm{I}_{T+1}\otimes\Sigma_\eta) + \hat{w}(\mathrm{I}_{T+1}\otimes\Sigma_\mu))^{-1}\right)\mathcal{K}\\
&= \mathcal{K}'\left(\mathrm{I}_{T+1}\otimes(\hat{z}\hat{w}\Sigma_\mu(\Sigma_\eta + \hat{w}\Sigma_\mu)^{-1})\right)\mathcal{K}\\
&= \mathcal{K}'\left(\mathrm{I}_{T+1}\otimes\mathrm{diag}\left(\hat{z}\hat{w}\sigma_{\mu,1}^2(\sigma_{\eta,1}^2 + \hat{w}\sigma_{\mu,1}^2)^{-1},\ldots,\hat{w}\sigma_{\mu,Z}^2(\sigma_{\eta,Z}^2 + \hat{w}\sigma_{\mu,Z}^2)^{-1}\right)\right)\mathcal{K}\\
&= \mathcal{K}'\left(\mathrm{I}_{T+1}\otimes\mathrm{diag}\left(\hat{r}_1,\ldots,\hat{r}_Z\right)\right)\mathcal{K},
\end{aligned}$$

which concludes the proof. □

Parts 1 and 2 of Proposition B.4. Given the result in Lemma B.9, we are left with taking the limit as $T\to\infty$ of the truncated problem. In particular, we have that

$$\lim_{T\to\infty} A\,(\mathrm{I}_m + \hat{R})\,\bar{\Delta}B'\left(B\bar{\Delta}B'\right)^{-1} = p\left(L; w, \alpha, \{r_i\}_{i=1}^Z\right), \quad \lim_{T\to\infty} A\mathcal{K}'\Omega_\mu\mathcal{K}A' = \mathbb{V}(\xi_t^\mu),$$

$$\lim_{T\to\infty}\,(A - h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}(A' - Bh') = \mathbb{V}(\xi_t^\mu - K_t^\mu), \quad \lim_{T\to\infty}(A - h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A' = \mathbb{COV}\left(\xi_t^\mu - K_t^\mu, \xi_t^\mu\right),$$

$$\lim_{T\to\infty}\frac{(A - h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A'}{A\mathcal{K}'\Omega_\mu\mathcal{K}A'} = 1 - \mathcal{S}^\mu.$$

Let $w \equiv \lim_{T\to\infty}\hat{w}$, and $r_i \equiv \lim_{T\to\infty}\hat{r}_i$, for $i\in\{1,\ldots,Z\}$. Then, we can show that

$$\begin{aligned}
r_i &= \lim_{T\to\infty}\frac{\hat{w}\tau_{\mu,i}}{1+\hat{w}\tau_{\mu,i}}\frac{\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}{1-\lambda\gamma A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\frac{(A - h'B)\,\mathcal{K}'\Omega_\mu\mathcal{K}A'}{A\mathcal{K}'\Omega_\mu\mathcal{K}A'}\\
&= \frac{w\tau_{\mu,i}}{1+w\tau_{\mu,i}}\frac{\lambda\gamma\mathbb{V}\left(\xi_t^\mu\right)}{1-\lambda\gamma\mathbb{V}\left(\xi_t^\mu\right)}\left(1-\mathcal{S}^\mu\right),
\end{aligned}$$

and

$$
\begin{aligned}
w &= \lim_{T\to\infty} \frac{1}{1-\lambda\left(1-\alpha\right)\left(A-h'B\right)\mathcal{K}'\left(\Omega_{\mu}+\frac{\lambda\gamma\Omega_{\mu}\mathcal{K}A'A\mathcal{K}'\Omega_{\mu}}{1-\lambda\gamma A\mathcal{K}'\Omega_{\mu}\mathcal{K}A'}\right)\mathcal{K}\left(A'-B'h\right)} \\
&= \lim_{T\to\infty} \frac{1}{1-\lambda\left(1-\alpha\right)\left(\left(A-h'B\right)\mathcal{K}'\Omega_{\mu}\mathcal{K}\left(A'-B'h\right)+\frac{\lambda\gamma\left(\left(A-h'B\right)\mathcal{K}'\Omega_{\mu}\mathcal{K}A'\right)\left(A\mathcal{K}'\Omega_{\mu}\mathcal{K}\left(A'-B'h\right)\right)}{1-\lambda\gamma A\mathcal{K}'\Omega_{\mu}\mathcal{K}A'}\right)} \\
&= \frac{1}{1-\lambda(1-\alpha)\left(\mathbb{V}\left(\xi_t^{\mu}-K_t^{\mu}\right)+\frac{\lambda\gamma\mathbb{V}(\xi_t^{\mu})^2(1-\mathcal{S}^{\mu})^2}{1-\lambda\gamma\mathbb{V}(\xi_t^{\mu})}\right)}.
\end{aligned}
$$

Part 3 of Proposition B.4. All the steps used in the proof of part 3 of Proposition 3.3 hold without change except for the last step. From those derivations we have that

$$
h_0 = \chi\lambda\left(A-h'B\right)\mathcal{K}'\Omega_{\mu}\mathcal{K}A'\left(\mathrm{I_u}+\frac{\lambda\gamma \mathrm{A}\mathcal{K}'\Omega_{\mu}\mathcal{K}\mathrm{A}'}{1-\lambda\gamma\mathrm{A}\mathcal{K}'\Omega_{\mu}\mathcal{K}\mathrm{A}'}\right)
$$

Taking the limit we get

$$
\mathcal{B} = \lim_{T\to\infty} h_0 = \chi\lambda\tau_{\mu}\mathbb{COV}(\xi_t^{\mu}-K_t^{\mu},\xi_t^{\mu})\left(1+\frac{\lambda\gamma\mathbb{V}(\xi_t^{\mu})}{1-\lambda\gamma\mathbb{V}(\xi_t^{\mu})}\right) = \frac{\chi\lambda\mathbb{V}\left(\xi_t^{\mu}\right)}{1-\lambda\gamma\mathbb{V}\left(\xi_t^{\mu}\right)}\left(1-\mathcal{S}^{\mu}\right),
$$

which completes the proof of part 3 of the proposition.

□

C Proofs of Other Results

Proof of Proposition 2.1. Following the same arguments used in the proof of Proposition 3.2, the optimal linear strategy, $g(x_i) \equiv s^* x_i + \mathcal{B}$, solves the following fixed point problem

$$s^* x_i + \mathcal{B} = \int_\mu \mathbb{E}^\mu [\xi | x_i] \hat{p}(\mu | x_i) \, \mathrm{d}\mu = \frac{\sigma_\xi^2}{\sigma_\xi^2 + \sigma_\epsilon^2} x_i + \frac{\sigma_\epsilon^2}{\sigma_\xi^2 + \sigma_\epsilon^2} \int_\mu \mu \, \hat{p}(\mu | x_i) \, \mathrm{d}\mu,$$

with

$$\begin{aligned} \hat{p}(\mu | x_i) &\propto \exp\left(\lambda \mathbb{E}^\mu \left[(s^* x_i + \mathcal{B} - \xi)^2 - \chi \xi\right]\right) p(x_i | \mu) \, p(\mu) \\ &\propto \exp\left(\lambda (1 - s^*)^2 \mu^2 + 2\lambda (s^* - 1) \mathcal{B} \mu - \chi \mu - \frac{(x_i - \mu)^2}{2\left(\sigma_\xi^2 + \sigma_\epsilon^2\right)} - \frac{1}{2\sigma_\mu^2} \mu^2\right). \end{aligned}$$

Mapping it into the kernel of a normal distribution yields

$$\mu \sim \mathcal{N}\left(\frac{\frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} x_i + 2\lambda (s^* - 1) \mathcal{B} - \lambda \chi}{\frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} - 2\lambda (1 - s^*)^2}, \frac{1}{\frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} - 2\lambda (1 - s^*)^2}\right),$$

which implies that

$$\int_\mu \mu \, \hat{p}(\mu | x_i) \, \mathrm{d}\mu = \frac{\frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} x_i + 2\lambda (s^* - 1) \mathcal{B} - \lambda \chi}{\frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} - 2\lambda (1 - s^*)^2}.$$

Matching coefficients leads to the following conditions

$$s^* = \frac{\sigma_\xi^2}{\sigma_\xi^2 + \sigma_\epsilon^2} + \frac{\sigma_\epsilon^2}{\sigma_\xi^2 + \sigma_\epsilon^2} \frac{\frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2}}{\frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} - 2\lambda (1 - s^*)^2},$$

and

$$\mathcal{B} = \frac{\sigma_\epsilon^2}{\sigma_\xi^2 + \sigma_\epsilon^2} \frac{2\lambda (s^* - 1) \mathcal{B} - \lambda \chi}{\frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_\xi^2 + \sigma_\epsilon^2} - 2\lambda (1 - s^*)^2}.$$

Solving for s^* and $\mathcal{B}$ leads to the expressions stated in the proposition. □

Proof of Corollary 3.1. Aggregating the individual best response in equation (3.6) leads to

$$K_t = (1 - \alpha) \overline{\mathcal{F}}_t^1 [\xi_t] + \alpha \overline{\mathcal{F}}_t^1 [K_t].$$

Iterating forward using the definitions of subjective higher-order expectations, it follows that

$$
\begin{aligned}
K_t &= (1-\alpha)\,\overline{\mathcal{F}}_t^1[\xi_t] + \alpha(1-\alpha)\,\overline{\mathcal{F}}_t^2[\xi_t] + \alpha^2\overline{\mathcal{F}}_t^2[K_t] \\
&= \cdots \\
&= (1-\alpha)\sum_{j=0}^{N}\alpha^j\overline{\mathcal{F}}_t^{j+1}[\xi_t] + \alpha^{N+1}\overline{\mathcal{F}}_t^{N+1}[K_t] \\
&= \cdots \\
&= (1-\alpha)\sum_{j=0}^{\infty}\alpha^j\overline{\mathcal{F}}_t^{j+1}[\xi_t],
\end{aligned}
$$

which completes the proof. □

Proof of Corollary 3.2. This result follows directly from the fact that $p(L;w,\alpha)$ permits a finite state representation. □

Proof of Proposition 3.4. Applying Proposition 3.3, we obtain

$$
\mathcal{F}_i[\xi] = \varsigma x_i - (1-\varsigma)\,\lambda\chi\sigma_\mu^2, \quad \text{with} \quad \varsigma \equiv \frac{(1+w)\,\sigma_\xi^2}{(1+w)\,\sigma_\xi^2 + \sigma_\epsilon^2}.
$$

Aggregating over i, it follows that

$$
\overline{\mathcal{F}}[\xi] = \varsigma\xi - (1-\varsigma)\,\lambda\chi\sigma_\mu^2.
$$

Applying the operator $\mathcal{F}_i$ to both sides and aggregating again yields,

$$
\overline{\mathcal{F}}^2[\xi] = \varsigma^2\xi - (1-\varsigma)(1+\varsigma)\,\lambda\chi\sigma_\mu^2.
$$

Iterating forward, it follows that

$$
\overline{\mathcal{F}}^m[\xi] = \varsigma^m\xi - (1-\varsigma)\sum_{k=0}^{m-1}\varsigma^k\lambda\chi\sigma_\mu^2 = \kappa_m\xi + \beta_m,
$$

with

$$
\kappa_m \equiv \varsigma^m, \quad \text{and} \quad \beta_m \equiv -(1-\varsigma)\sum_{k=0}^{m-1}\varsigma^k\lambda\chi\sigma_\mu^2.
$$

Therefore, we have that

$$
\beta_m = \beta_{m-1} - (1-\varsigma)\,\kappa_{m-1}\lambda\chi\sigma_\mu^2 = \beta_{m-1} + (\kappa_m - \kappa_{m-1})\,\lambda\chi\sigma_\mu^2,
$$

which completes the proof of Part 1. Moreover, combing equation (3.7) with the fact that $\overline{\mathcal{F}}^m[\xi] = \kappa_m\xi + \beta_m$ leads to

$$
K = (1-\alpha)\sum_{m=0}^{\infty}\alpha^m\kappa_m\,\xi + (1-\alpha)\sum_{m=0}^{\infty}\alpha^m\beta_m,
$$

which completes the proof of Part 3.

To establish Part 2 notice that, from Proposition 3.3, we have

$$w = \left[\frac{1}{\tau_\mu} - \lambda\left(1-\alpha\right)\mathbb{V}\left(\xi - K\right)\right]^{-1},$$

which, differentiating with respect to α, yields

$$\frac{\mathrm{d}w}{\mathrm{d}\alpha} = \lambda w^2\left[\left(1-\alpha\right)\frac{\mathrm{d}\mathbb{V}\left(\xi-K\right)}{\mathrm{d}\alpha} - \mathbb{V}\left(\xi-K\right)\right].$$

Since

$$\mathbb{V}\left(\xi-K\right) = \left(\frac{\sigma_\epsilon^2}{\left(1+w\right)\left(1-\alpha\right)\sigma_\xi^2 + \sigma_\epsilon^2}\right)^2 \sigma_\xi^2,$$

it follows that

$$\frac{\mathrm{d}\mathbb{V}\left(\xi-K\right)}{\alpha} = 2\left(\frac{\sigma_\xi^2}{\left(1+w\right)\left(1-\alpha\right)\sigma_\xi^2 + \sigma_\epsilon^2}\right)\mathbb{V}\left(\xi-K\right)\left(w - \left(1-\alpha\right)\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right),$$

and, therefore,

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}\alpha} &= \lambda w^2\mathbb{V}\left(\xi-K\right)\left[2\left(1-\alpha\right)\left(\frac{\sigma_\xi^2}{\left(1+w\right)\left(1-\alpha\right)\sigma_\xi^2 + \sigma_\epsilon^2}\right)\left(w - \left(1-\alpha\right)\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right) - 1\right] \\ &= \frac{\lambda w^2\mathbb{V}\left(\xi-K\right)\left(\left(w-1\right)\left(1-\alpha\right)\sigma_\xi^2 - \sigma_\epsilon^2\right)}{\left[1+w+2\lambda\left(1-\alpha\right)w^2\mathbb{V}\left(\xi-K\right)\right]\left(1-\alpha\right)\sigma_\xi^2 + \sigma_\epsilon^2}.\end{aligned}$$

Then, since, in the limit as α increases to 1, we have that $w \to \tau_\mu$, and $\mathbb{V}(\xi - K) \to \mathbb{V}(\xi)$, it follows that

$$\lim_{\alpha\to1^-}\frac{\mathrm{d}w}{\mathrm{d}\alpha} = -\lambda\tau_\mu^2\mathbb{V}(\xi) < 0.$$

On the other hand, notice that

$$\operatorname{sgn}\left[\lim_{\alpha\to0^+}\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right] = \operatorname{sgn}\left[\left(w-1\right)\sigma_\xi^2 - \sigma_\epsilon^2\right],$$

so that, since $w \geq \tau_\mu$, we have that

$$\tau_\mu > \frac{\sigma_\xi^2 + \sigma_\epsilon^2}{\sigma_\xi^2} \quad\Rightarrow\quad \lim_{\alpha\to0^+}\frac{\mathrm{d}w}{\mathrm{d}\alpha} > 0.$$

Hence, w is non-monotonic in α if τ_μ is large enough. □

Proof of Proposition 3.5. It follows from Proposition 3.3 that, when $\gamma = 0$,

$$\mathcal{B} = \chi\lambda\tau_\mu \mathbb{V}(\xi_t)(1 - \mathcal{S}).$$

Therefore, to prove that $|\mathcal{B}|$ is increasing in α, it is sufficient to prove that the sensitivity $\mathcal{S}$ is decreasing in α. Since, by definition

$$\mathcal{S} = \frac{\mathbb{COV}(K_t, \xi_t)}{\mathbb{V}(\xi_t)},$$

with $\mathbb{V}(\xi_t)$ independent of α, it is sufficient to show that

$$\frac{\mathrm{d}\mathbb{COV}(K_t, \xi_t)}{\mathrm{d}\alpha} < 0.$$

Following the notation of the truncated economy introduced in the proof of Proposition 3.3, we have that

$$\mathbb{COV}(K_t, \xi_t) = h' B \Lambda \Omega A',$$

with h denoting the optimal forecasting rule

$$h = A\bar{\Omega}B' \left(B\bar{\Omega}B'\right)^{-1}, \quad \text{with} \quad \bar{\Omega} = (1+w)\Lambda\Omega + (1-\alpha)^{-1}(\mathrm{I}_m - \Lambda)\Omega.$$

Since Ω is diagonal, we can rewrite h as

$$h = A\hat{\Omega}B' \left(B\hat{\Omega}B'\right)^{-1}, \quad \text{with} \quad \hat{\Omega} = \Lambda\Omega + m_\alpha (\mathrm{I}_m - \Lambda)\Omega, \quad \textit{and} \quad m_\alpha \equiv [(1-\alpha)(1+w)]^{-1}.$$

It follows that

$$\begin{aligned}
\frac{\mathrm{d}\mathbb{COV}(K_t, \xi_t)}{\mathrm{d}\alpha} &= A\Omega\Lambda B' \frac{\mathrm{d}\left(B\hat{\Omega}B'\right)^{-1}}{\mathrm{d}\alpha} B\Lambda\Omega A' \\
&= -A\Omega\Lambda B' \left(B\hat{\Omega}B'\right)^{-1} B \frac{\mathrm{d}\hat{\Omega}}{\mathrm{d}\alpha} B' \left(B\hat{\Omega}B'\right)^{-1} B\Lambda\Omega A' \\
&= -\left(z'(\mathrm{I}_m - \Lambda)\Omega z\right) m_\alpha^2 \left[(1+w) - (1-\alpha)\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right],
\end{aligned}$$

where z is a column vector,

$$z \equiv B' \left(B\hat{\Omega}B'\right)^{-1} B\Lambda\Omega A'.$$

Since $(\mathrm{I}_m - \Lambda)\Omega$ is positive semi-definite, it follows that

$$\mathrm{sgn}\left[\frac{\mathrm{d}\mathbb{COV}(K_t, \xi_t)}{\mathrm{d}\alpha}\right] = -\mathrm{sgn}\left[(1+w) - (1-\alpha)\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right].$$

Further, notice that since $w \geq \tau_\mu$ and $\lim_{\alpha \to 1^-} w = \tau_\mu$, we have that the $\lim_{\alpha \to 1^-} \mathrm{d}w/\mathrm{d}\alpha$ is bounded and, therefore,

$$\lim_{\alpha \to 1^-} \frac{\mathrm{d}\mathbb{COV}\left(K_t, \xi_t\right)}{\mathrm{d}\alpha} < 0.$$

Finally, for a contradiction, suppose there exists some $\alpha \in [0, 1)$ such that $\mathrm{d}\mathbb{COV}\left(K_t, \xi_t\right)/\mathrm{d}\alpha > 0$. It follows from the intermediate value theorem and the continuity of $\mathrm{d}\mathbb{COV}\left(K_t, \xi_t\right)/\mathrm{d}\alpha$ that there must exist some $\alpha_\dagger$ such that

$$\left.\frac{\mathrm{d}\mathbb{COV}\left(K_t, \xi_t\right)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger} = 0 \;\Rightarrow\; \left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger} = \frac{1+w_\dagger}{1-\alpha_\dagger} > 0,$$

where $w_\dagger$ denotes w evaluated at $\alpha_\dagger$. With $\gamma = 0$, Proposition 3.3 implies that

$$w = \left[\frac{1}{\tau_\mu} - \lambda\left(1-\alpha\right)\mathbb{V}\left(\xi_t - K_t\right)\right]^{-1},$$

and it follows that

$$\frac{\mathrm{d}w}{\mathrm{d}\alpha} = -\lambda w^2\left[\mathbb{V}\left(\xi_t - K_t\right) - \left(1-\alpha\right)\frac{\mathrm{d}\mathbb{V}\left(\xi_t - K_t\right)}{\mathrm{d}\alpha}\right].$$

Using the fact that, similarly to $\mathbb{COV}\left(\xi_t, K_t\right)$, $\mathbb{V}\left(\xi_t - K_t\right)$ depends on α only through m_α, we have that

$$\left.\frac{\mathrm{d}\mathbb{V}\left(\xi_t - K_t\right)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger} = \left.\frac{\mathrm{d}\mathbb{V}\left(\xi_t - K_t\right)}{\mathrm{d}m_\alpha}\frac{\mathrm{d}m_\alpha}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger} = \left.\frac{\mathrm{d}\mathbb{V}\left(\xi_t - K_t\right)}{\mathrm{d}m_\alpha}m_\alpha^2\left[\left(1+w\right) - \left(1-\alpha\right)\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right]\right|_{\alpha=\alpha_\dagger} = 0,$$

and, therefore,

$$\left.\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_\dagger} = -\lambda w^2\mathbb{V}\left(\xi_t - K_t\right) < 0,$$

which yields the desired contradiction. □

Proof of Lemma 4.1. We start by characterizing the zero-inflation steady state. From the budget constraint of household i, we have that

$$C_{i,g,t+1} = \frac{Y_g - C_{i,g,t}}{1+\pi_{t+1}}.$$

Substituting $C_{i,g,t+1}$ into the utility function $U\left(C_{i,g,t}, C_{i,g,t+1}\right)$ yields

$$U\left(C_{i,g,t}, \pi_{t+1}\right) = \frac{C_{i,g,t}^{1-\nu}}{1-\nu} + \beta\frac{\left(\frac{Y_g - C_{i,g,t}}{1+\pi_{t+1}}\right)^{1-\nu}}{1-\nu}.$$

The Euler equation in the zero-inflation steady state implies that

$$\bar{C}_g^{-\nu} - \beta\left(Y_g - \bar{C}_g\right)^{-\nu} = 0. \tag{C.1}$$

Let $c_{i,g,t}$ be the log-deviation from the zero-inflation steady state, that is

$$c_{i,g,t} \equiv \log C_{i,g,t} - \log \bar{C}_g.$$

The quadratic approximation of $U(C_{i,g,t}, \pi_{t+1})$ around the zero-inflation steady state leads to

$$\begin{aligned}
U(C_{i,g,t}, \pi_{t+1}) \approx\ & \mathcal{Q}(\hat{c}_t, \pi_{t+1}) \\
\equiv\ & const - \bar{C}_g^{1-\nu} \left(\frac{Y - \bar{C}_g}{\bar{C}_g}\right) \pi_{t+1} + (1-\nu) \bar{C}_g^{1-\nu} c_{i,g,t} \pi_{t+1} \\
& + \frac{1}{2}(1-\nu) \bar{C}_g^{1-\nu} \left(\frac{Y_g - \bar{C}_g}{\bar{C}_g}\right) \pi_{t+1}^2 - \frac{1}{2} \nu \bar{C}_g^{1-\nu} \left[1 + \left(\frac{Y_g - \bar{C}_g}{\bar{C}_g}\right)^{-1}\right] c_{i,g,t}^2 \\
=\ & const - \bar{C}_g^{1-\nu} \left(\frac{Y_g - \bar{C}_g}{\bar{C}_g}\right) \pi_{t+1} + \frac{1}{2}(1-\nu) \bar{C}_g^{1-\nu} \left(\frac{Y_g - \bar{C}_g}{\bar{C}_g} + \frac{1-\nu}{\nu} \frac{Y_g - \bar{C}_g}{Y_g}\right) \pi_{t+1}^2 \\
& - \frac{1}{2} \nu \bar{C}_g^{1-\nu} \frac{Y_g}{Y_g - \bar{C}_g} \left(c_{i,g,t} - \frac{1-\nu}{\nu} \frac{Y_g - \bar{C}_g}{Y_g} \pi_{t+1}\right)^2.
\end{aligned}$$

Given subjective beliefs $\mathcal{F}_{i,g,t}[\cdot]$, the optimal consumption must be proportional to the households subjective expectation about inflation:

$$\begin{aligned}
c_{i,g,t} =& \frac{1-\nu}{\nu} \frac{Y_g - \bar{C}_g}{Y_g} \mathcal{F}_{i,g,t}[\pi_{t+1}] \\
=& \frac{\beta^{1/\nu}}{1 + \beta^{1/\nu}} \mathcal{F}_{i,g,t}[\pi_{t+1}],
\end{aligned}$$

where the last equality directly follows equation (C.1).

In the smooth model of ambiguity, similarly to the proof of Proposition 3.2, it can be shown that

$$c_{i,g,t} = \frac{1-\nu}{\nu} \frac{Y_g - \bar{C}_g}{Y_g} \int_{\mu^t} \mathbb{E}^{\mu^t}[\pi_{t+1} | \mathcal{I}_{i,g,t}] \hat{p}\left(\mu^t | \mathcal{I}_{i,g,t}\right) \mathrm{d}\mu^t,$$

where the distorted posterior $\hat{p}(\mu^t | \mathcal{I}_{i,g,t})$ is such that

$$\hat{p}\left(\mu^t | \mathcal{I}_{i,g,t}\right) \propto \exp\left(-\lambda \mathbb{E}^{\mu^t}[\mathcal{Q}(\hat{c}_t, \pi_{t+1})]\right).$$

Let the subjective belief of the household be such that

$$\mathcal{F}_{i,g,t}[\cdot] \equiv \int_{\mu^t} \mathbb{E}^{\mu^t}[\cdot | \mathcal{I}_{i,g,t}] \hat{p}\left(\mu^t | \mathcal{I}_{i,g,t}\right) \mathrm{d}\mu^t,$$

then, it follows that

$$c_{i,g,t} = \frac{\beta^{1/\nu}}{1 + \beta^{1/\nu}} \mathcal{F}_{i,g,t}[\pi_{t+1}],$$

which yields equation (4.2). Substituting $c_{i,g,t}$ into $\mathcal{Q}(\hat{c}_t, \pi_{t+1})$, leads to equation (4.3) with

$$\chi_g \equiv Y_g^{1-\nu} \frac{\beta^{1/\nu}}{\left(1+\beta^{1/\nu}\right)^{1-\nu}}$$
$$\gamma_g \equiv \frac{1}{2} Y_g^{1-\nu} \frac{(\nu-1)\,\beta^{1/\nu}}{\left(1+\beta^{1/\nu}\right)^{1-\nu}} \frac{1+\nu\beta^{1/\nu}}{\nu\left(1+\beta^{1/\nu}\right)}$$
$$\delta_g \equiv \frac{1}{2} Y_g^{1-\nu} \frac{(1-\nu)^2\,\beta^{1/\nu}}{\nu\left(1+\beta^{1/\nu}\right)^{2-\nu}}.$$

Notice that δ_g, χ_g, and γ_g are all proportional to $Y_g^{\nu-1}$. Moreover, when $\nu > 1$, they are all positive and decreasing in Y_g. □

The following lemma is used in the proof of the next propositions.

Lemma C.1 (Kalman filter for AR(1))**.** *Given a state equation*

$$\xi_t = \rho \xi_{t-1} + \nu_t, \quad \textit{with} \quad \nu_t \sim \mathcal{N}\left(0, \sigma_\nu^2\right),$$

and an observation equation

$$x_t = \xi_t + u_t, \quad \textit{with} \quad u_t \sim \mathcal{N}\left(0, \sigma_u^2\right),$$

the steady-state Kalman gain is given by

$$\kappa = \frac{1}{2\rho}\left(\rho - \frac{\sigma_u^2 + \sigma_\nu^2}{\rho\sigma_u^2} - \sqrt{\left(\rho - \frac{\sigma_u^2+\sigma_\nu^2}{\rho\sigma_u^2}\right)^2 + 4\frac{\sigma_\nu^2}{\sigma_u^2}}\right),$$

and the updating rule for the Bayesian forecast follows

$$\mathbb{E}_t\left[\xi_{t+1}\right] = \rho\,(1-\kappa)\,\mathbb{E}_{t-1}\left[\xi_t\right] + \rho\kappa x_t.$$

Proof of Proposition 4.1. Cosider Lemma C.1 with $\xi_t = \pi_t$, $\sigma_\nu^2 = \sigma_\eta^2$, and $\sigma_u^2 = \sigma_\varepsilon^2$, and define $\omega \equiv \rho\,(1-\kappa)$. Since every agent i in every group g has the same information structure with signals given by

$$x_{i,g,t} = \pi_t + \varepsilon_{i,g,t}, \quad \text{with} \quad \varepsilon_{i,g,t} \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right),$$

it immediately follows from Lemma C.1 that

$$\mathbb{E}_{i,g,t}\left[\pi_{t+1}\right] = \omega \mathbb{E}_{i,g,t-1}\left[\pi_t\right] + (\rho - \omega)\,x_{i,g,t},$$

and

$$\omega = \frac{1}{2}\left(\rho + \frac{\sigma_\varepsilon^2+\sigma_\eta^2}{\rho\sigma_\varepsilon^2} - \sqrt{\left(\rho + \frac{\sigma_\varepsilon^2+\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4}\right).$$

Integrating the updating rule for the forecast, we have that

$$\int \mathbb{E}_{i,g,t}\left[\pi_{t+1}\right] = \omega \int \mathbb{E}_{i,g,t-1}\left[\pi_t\right] + (\rho-\omega)\int x_{i,g,t}$$

and, therefore,

$$\overline{\mathbb{E}}_{g,t}\left[\pi_{t+1}\right] = \omega\overline{\mathbb{E}}_{g,t-1}\left[\pi_t\right] + (\rho-\omega)\,\pi_t,$$

which can be rewritten as

$$\overline{\mathbb{E}}_{g,t}\left[\pi_{t+1}\right] = \frac{\rho-\omega}{1-\omega L}\pi_t.$$

The average forecast error is, then, given by

$$\begin{aligned}
\pi_{t+1} - \overline{\mathbb{E}}_{g,t}\left[\pi_{t+1}\right] &= \pi_{t+1} - \frac{\rho-\omega}{1-\omega L}\pi_t \\
&= \frac{\eta_{t+1}}{1-\rho L} - \frac{\rho-\omega}{1-\omega L}\frac{L\eta_{t+1}}{1-\rho L} \\
&= \frac{\eta_{t+1}}{1-\omega L},
\end{aligned}$$

which concludes the proof. □

Proof of Proposition 4.2. It follows from Proposition 3.3 that

$$\mathcal{F}_{i,g,t}\left[\pi_{t+1}\right] = (1+r_g)\,\mathbb{E}_{i,g,t}\left[\pi_{t+1}\right] + \mathcal{B}_g$$

where $\mathbb{E}_{i,g,t}\left[\pi_{t+1}\right]$ denotes the periot-t Bayesian forecast of π_{t+1} of agent i in group g given the $(w_g, 0)$-modified information structure (notice that here $\alpha = 0$). Thus, setting $\xi_t = \pi_t$, $\sigma_\nu^2 = (1+w)\,\sigma_\eta^2$, and $\sigma_u^2 = \sigma_\varepsilon^2$, it follows from Lemma C.1 that

$$\mathbb{E}_{i,g,t}\left[\pi_{t+1}\right] = \rho\,(1-\kappa_g)\,\mathbb{E}_{i,g,t-1}\left[\pi_t\right] + \rho\kappa_g x_{i,g,t},$$

with

$$\kappa_g = \frac{1}{2\rho}\left(\left(\rho - \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right) - \sqrt{\left(\rho - \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 + 4\frac{(1+w_g)\,\sigma_\eta^2}{\sigma_\varepsilon^2}}\right).$$

It follows that

$$(1+r_g)\,\mathbb{E}_{i,g,t}\left[\pi_{t+1}\right] + \mathcal{B}_g = \rho\,(1-\kappa_g)\left((1+r_g)\,\mathbb{E}_{i,g,t-1}\left[\pi_t\right] + \mathcal{B}_g\right) + (1+r_g)\,\rho\kappa_g x_{i,g,t} - \rho\,(1-\kappa_g)\,\mathcal{B}_g + \mathcal{B}_g$$

and, therefore,

$$\mathcal{F}_{i,g,t}\left[\pi_{t+1}\right] = \rho\,(1-\kappa_g)\,\mathcal{F}_{i,g,t-1}\left[\pi_t\right] + (1+r_g)\,\rho\kappa_g x_{i,g,t} + (1-\rho\,(1-\kappa_g))\,\mathcal{B}_g.$$

Defining $\vartheta_g \equiv \rho\,(1-\kappa_g)$, we obtain

$$\mathcal{F}_{i,g,t}\left[\pi_{t+1}\right] = \vartheta_g\mathcal{F}_{i,g,t-1}\left[\pi_t\right] + (1+r_g)\,(\rho-\vartheta_g)\,x_{i,g,t} + (1-\vartheta_g)\,\mathcal{B}_g,$$

with

$$\vartheta_g = \frac{1}{2}\left(\rho + \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2} - \sqrt{\left(\rho + \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4}\right).$$

Integrating the updating rule for the forecast, we have that

$$\int \mathcal{F}_{i,g,t}\left[\pi_{t+1}\right] = \vartheta_g \int \mathcal{F}_{i,g,t-1}\left[\pi_t\right] + (1+r_g)\,(\rho - \vartheta_g) \int x_{i,g,t} + (1-\vartheta_g)\,\mathcal{B}_g$$

and, therefore,

$$\overline{\mathcal{F}}_{g,t}\left[\pi_{t+1}\right] = \vartheta_g \overline{\mathcal{F}}_{g,t-1}\left[\pi_t\right] + (1+r_g)\,(\rho - \vartheta_g)\,\pi_t + (1-\vartheta_g)\,\mathcal{B}_g,$$

which can be rewritten as

$$\overline{\mathcal{F}}_{g,t}\left[\pi_{t+1}\right] = \frac{(1+r_g)\,(\rho-\vartheta_g)\,\pi_t}{1-\vartheta_g L} + \mathcal{B}_g.$$

The average forecast error is, then, given by

$$\begin{aligned} \pi_{t+1} - \overline{\mathcal{F}}_{g,t}\left[\pi_{t+1}\right] &= \pi_{t+1} - \frac{(1+r_g)\,(\rho-\vartheta_g)\,\pi_t}{1-\vartheta_g L} - \mathcal{B}_g \\ &= \frac{(1+r_g)\,\eta_{t+1}}{1-\vartheta_g L} - \frac{r_g}{1-\rho L}\eta_{t+1} - \mathcal{B}_g. \end{aligned}$$

The fact that $r_g > 0$, $w_g > 0$, and $\mathcal{B}_g > 0$ follows immediately from Proposition 3.3 together with the fact that $\delta_g > 0$, $\chi_g > 0$, and $\gamma_g > 0$ established in Lemma 4.1 and that, by assumption, $\lambda > 0$ and $\sigma_\mu^2 > 0$. Finally, to see that $\vartheta_g < \omega$ notice that, from the triangle inequality, we have that

$$\sqrt{\left(\rho + \frac{\sigma_\varepsilon^2 + \sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4} + \sqrt{\left(\frac{w_g\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2} < \sqrt{\left(\rho + \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4},$$

so that

$$\frac{w_g\sigma_\eta^2}{\rho\sigma_\varepsilon^2} - \sqrt{\left(\rho + \frac{\sigma_\varepsilon^2 + (1+w_g)\,\sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4} < -\sqrt{\left(\rho + \frac{\sigma_\varepsilon^2 + \sigma_\eta^2}{\rho\sigma_\varepsilon^2}\right)^2 - 4}.$$

Adding $\rho + \frac{\sigma_\varepsilon^2 + \sigma_\eta^2}{\rho\sigma_\varepsilon^2}$ and dividing by 2 yields the result. □

Proof of Proposition 5.1. Under rational expectations, the optimal inflation forecast is such that

$$\mathcal{F}_i\left[\pi\right] = \mathbb{E}_i\left[(1-\alpha)\,\pi^* + \alpha\overline{\mathcal{F}}\left[\pi\right]\right].$$

It follows from the the equivalence result in Huo and Pedroni (2020), that the optimal forecast is given by

$$\mathcal{F}_i\left[\pi\right] = \frac{\sigma_\pi^2}{\sigma_\pi^2 + (1-\alpha)^{-1}\,\sigma_\epsilon^2}\,x_i.$$

Aggregating, we obtain

$$\overline{\mathcal{F}}[\pi]=\frac{\sigma_\pi^2}{\sigma_\pi^2+(1-\alpha)^{-1}\sigma_\epsilon^2}\pi^*.$$

Plugging this into the time-invariant inflation policy rule (5.1) completes the proof. □

Proof of Proposition 5.2. To ease notation, let

$$k_i\equiv\mathcal{F}_i[\pi],\quad\text{and}\quad K=\overline{\mathcal{F}}[\pi].$$

Plugging (5.1) into the utility function of the agent results in

$$\begin{aligned}u(k_i,K,\pi^*)&=-(k_i-(1-\alpha)\pi^*-\alpha K)^2-\chi((1-\alpha)\pi^*+\alpha K)\\&=-\left[(1-\alpha)(k_i-\pi^*)^2+\alpha(k_i-K)^2\right]-(1-\alpha)\chi\pi^*+\alpha(1-\alpha)(K-\pi^*)^2-\alpha\chi K.\end{aligned}$$

This is an inefficient economy, so we use Proposition B.3 to characterize the optimal forecasts. Let

$$\begin{aligned}&\lambda_{\text{ineff.}}\equiv 2\lambda,\quad\alpha_{\text{ineff.}}\equiv\alpha,\quad\gamma_{\text{ineff.}}\equiv 0,\quad\chi_{\text{ineff.}}\equiv\frac{1}{2}(1-\alpha)\chi,\\&\psi_{\text{ineff.}}\equiv-\alpha(1-\alpha),\quad\phi_{\text{ineff.}}\equiv 0,\quad\text{and}\quad\varphi_{\text{ineff.}}\equiv\frac{1}{2}\alpha\chi,\end{aligned}$$

where parameters with a subscript "ineff." correspond to the ones in the setup of Proposition B.3. It follows that

$$w=\frac{\tau_\mu}{1-2\lambda(1-\alpha)^2\tau_\mu\mathbb{V}(\pi-K)},\quad\text{and}\quad r=0,$$

where $\tau_\mu\equiv\sigma_\mu^2/\sigma_\pi^2$ is the normalized amount of ambiguity. Moreover, the bias is given by

$$\mathcal{B}=\lambda(1-\alpha)\chi\tau_\mu\mathbb{V}(\pi)(1-\mathcal{S})+\lambda\alpha\chi\tau_\mu\left[\mathbb{V}(\pi)(1-\mathcal{S})-\mathbb{V}(\pi-K)\right].$$

Using $\mathbb{V}(\pi)=\sigma_\pi^2$ and $\mathbb{V}(\pi-K)=(1-\mathcal{S})^2\sigma_\pi^2$, we obtain the desired expressions for sensitivity $\mathcal{S}$ and bias $\mathcal{B}$. Finally, the implied inflation policy directly follows from equation (5.1), which completes the proof. □

Proof of Proposition 5.3. Since the loss function is continuous in σ_μ^2, it is sufficient to show that

$$\left.\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\sigma_\mu^2}\right|_{\sigma_\mu^2=0}<0.$$

Fist notice that

$$\mathcal{L}=\frac{\omega}{\alpha}\left[(1-\mathcal{R})^2\sigma_\pi^2+\mathcal{C}^2\right]\;\Rightarrow\;\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\sigma_\mu^2}=\frac{2\omega}{\alpha}\left[-(1-\mathcal{R})\sigma_\pi^2\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\sigma_\mu^2}+\mathcal{C}\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\sigma_\mu^2}\right].$$

If $\sigma_\mu^2 = 0$, it is optimal to set $\mathcal{R} < 1$ and $\mathcal{C} = 0$, so that it is sufficient to show that

$$\left.\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\sigma_\mu^2}\right|_{\sigma_\mu^2=0} > 0,$$

or, equivalently,

$$\left.\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\sigma_\mu^2}\right|_{\sigma_\mu^2=0} > 0,$$

since $\mathcal{R} = 1 - \alpha + \alpha\mathcal{S}$. Further, notice that sensitivity $\mathcal{S}$ depends on σ_μ^2 only through w and is monotonically increasing in w, it is then sufficient to show that

$$\left.\frac{\mathrm{d}w}{\mathrm{d}\sigma_\mu^2}\right|_{\sigma_\mu^2=0} > 0.$$

This, in turn, follows from the fact that $w = 0$ if $\sigma_\mu^2 = 0$, and $w > 0$ for any $\sigma_\mu^2 > 0$. □

Proof of Proposition 5.4. The optimal inflation forecast must satisfy

$$\mathcal{F}_i[\pi] = (1-\alpha)\,\mathcal{F}_i[\pi^*] + \alpha\mathcal{F}_i\left[\overline{\mathcal{F}}[\pi]\right].$$

With heterogeneous priors, the belief system of agent i is such that

$$\mathcal{F}_i[\pi^*] = \mathbb{E}_i[\pi^*] = \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right) x_i, \quad \text{and}$$
$$\mathcal{F}_i[\mathcal{F}_j[\pi^*]] = \mathcal{F}_i\left[\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\,x_j + \frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\,\mathcal{B}\right] = \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^2 x_i + \left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B}.$$

It follows that

$$\mathcal{F}_i\left[\overline{\mathcal{F}}[\pi^*]\right] = \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^2 x_i + \left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B}, \quad \text{and}$$
$$\mathcal{F}_i\left[\mathcal{F}_j\left[\overline{\mathcal{F}}[\pi^*]\right]\right] = \mathcal{F}_i\left[\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^2 x_j + \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B} + \left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B}\right],$$

and, therefore,

$$\mathcal{F}_i\left[\overline{\mathcal{F}}^2[\pi^*]\right] = \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^3 x_i + \left(\sum_{s=0}^{1}\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^s\right)\left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B}.$$

Continuing to iterate forwards, we obtain that, for all $k \geq 1$,

$$\mathcal{F}_i\left[\overline{\mathcal{F}}^k\left[\pi^*\right]\right] = \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^k \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right) x_i + \left(\sum_{s=0}^{k-1}\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^s\right)\left(\frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)\mathcal{B}$$
$$= \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^k \left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right) x_i + \left(1-\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^k\right)\mathcal{B}.$$

Notice that the optimal forecast of agent i can be expressed as a weighted sum of higher-order beliefs,

$$\mathcal{F}_i\left[\pi\right] = (1-\alpha)\,\mathcal{F}_i\left[\pi^*\right] + (1-\alpha)\sum_{k=1}^{\infty}\alpha^k\mathcal{F}_i\left[\overline{\mathcal{F}}^k\left[\pi^*\right]\right]$$
$$= (1-\alpha)\left(\sum_{k=0}^{\infty}\alpha^k\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^k\right)\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right) x_i + (1-\alpha)\sum_{k=1}^{\infty}\alpha^k\left(1-\left(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}\right)^k\right)\mathcal{B}$$
$$= \mathcal{S}^{\mathrm{RE}}\,x_i + \alpha\left(1-\mathcal{S}^{\mathrm{RE}}\right)\mathcal{B},$$

where $\mathcal{S}^{\mathrm{RE}} \equiv \frac{\sigma_\pi^2}{\sigma_\pi^2+(1-\alpha)^{-1}\sigma_\epsilon^2}$ denotes the sensitivity under rational expectations.

From the inflation policy in equation (5.1), it follows that

$$\mathcal{R} = 1-\alpha+\alpha\mathcal{S}^{\mathrm{RE}} = \mathcal{R}^{\mathrm{RE}}, \quad \text{and} \quad \mathcal{C} = \alpha\left(\alpha-\alpha\mathcal{S}^{\mathrm{RE}}\right)\mathcal{B} = \alpha\left(1-\mathcal{R}^{\mathrm{RE}}\right)\mathcal{B}.$$

Finally, the social loss function is given by

$$\mathcal{L} = \frac{\omega}{\alpha}\left[\left(1-\mathcal{R}\right)^2\sigma_\pi^2+\mathcal{C}^2\right],$$

which is increasing in $\mathcal{B}$ since $\mathcal{C} = \alpha\left(1-\mathcal{R}^{\mathrm{RE}}\right)\mathcal{B}$. □

D Uniqueness and Linearity of Optimal Strategies without Strategic Interactions

In this Appendix, we prove that in the absence of strategic interactions, the optimal strategy is unique and linear in signals. It is worth noting that the uniqueness of the optimal strategy only requires concavity of the utility function $u(\cdot)$ and the $\phi(\cdot)$ function (Lemma D.1). Linearity, on the other hand, requires $u(\cdot)$ to be quadratic, $\phi(\cdot)$ to be of CAAA form, and the information structure to be Gaussian (Lemma D.2).

We base our analysis on the truncated economy outlined in the proof of Proposition 3.3, while shutting down strategic interactions by suppressing the dependence of the utility function on the aggregate action K:

$$\max_{\{k(x_i)\}} \int_\mu \phi\left(\mathbb{E}^\mu\left[u\left(k(x_i),\theta\right)\right]\right) p(\mu)\mathrm{d}\mu.$$

Agent i must choose an ex-ante strategy $k(x_i)$, a function of their entire history of private information, x_i.

Lemma D.1. *Without strategic interactions, there is a unique optimal strategy $k_i = g\left(x_i\right)$.*

Proof. To simplify notation, denote

$$\mathcal{W}\left(f\right) = \int_\mu \phi\left(\mathbb{E}^\mu\left[u\left(f,\theta\right)\right]\right) p\left(\mu\right)\mathrm{d}\mu, \quad \text{and} \quad \bar{\mathcal{W}} = \max_f \mathcal{W}\left(f\right).$$

Suppose there are at least two strategies $g_1\left(x_i\right)$ and $g_2\left(x_i\right)$ with $g_1 \neq g_2$ both achieving the optimum, that is, $\mathcal{W}\left(g_1\right) = \mathcal{W}\left(g_2\right) = \bar{\mathcal{W}}$. Consider an alternative strategy $h = \frac{g_1+g_2}{2}$. It follows that

$$\begin{aligned}
\mathcal{W}\left(h\right) &> \int_\mu \phi\left(\mathbb{E}^\mu\left[\frac{1}{2}u\left(g_1,\theta\right) + \frac{1}{2}u\left(g_2,\theta\right)\right]\right) p\left(\mu\right)\mathrm{d}\mu \\
&= \int_\mu \phi\left(\frac{1}{2}\mathbb{E}^\mu\left[u\left(g_1,\theta\right)\right] + \frac{1}{2}\mathbb{E}^\mu\left[u\left(g_2,\theta\right)\right]\right) p\left(\mu\right)\mathrm{d}\mu \\
&> \int_\mu \left(\frac{1}{2}\phi\left(\mathbb{E}^\mu\left[u\left(g_1,\theta\right)\right]\right) + \frac{1}{2}\phi\left(\mathbb{E}^\mu\left[u\left(g_1,\theta\right)\right]\right)\right) p\left(\mu\right)\mathrm{d}\mu \\
&= \frac{1}{2}\mathcal{W}\left(g_1\right) + \frac{1}{2}\mathcal{W}\left(g_2\right) = \bar{\mathcal{W}},
\end{aligned}$$

where the first and second inequalities use the concavity of u and ϕ, respectively. The condition $\mathcal{W}\left(h\right) > \bar{\mathcal{W}}$ contradicts the assumption that g_1 and g_2 are both optimal strategies. As a result, it must be the case that there exists a unique optimal strategy g. □

Lemma D.2. *If $u\left(\cdot\right)$ is quadratic, $\phi\left(\cdot\right)$ takes the CAAA form, and the information structure is Gaussian, the optimal strategy is unique and linear in signals, i.e., there exist unique h' and h_0 such that*

$$k_i = g\left(x_i\right) = h'x_i + h_0.$$

Proof. Notice that the economy under consideration is a special case of our model in Section 3 in which there are no strategic interactions, i.e., $\alpha = 0$. Then, invoking Proposition 3.1, we know that a linear optimal strategy exists. Combining this with the uniqueness result of Lemma D.1 completes the proof. □

E Robust Preferences: Derivations and Proofs

Lemma E.1. *Taking the law of motion of K_t as given, individual i's best response satisfies*

$$k_{it} = (1-\alpha)\,\mathcal{F}_{it}[\xi_t] + \alpha \mathcal{F}_{it}[K_t],$$

where $\mathcal{F}_{it}[\cdot]$ denotes agent i's subjective expectation, such that for any random variable X,

$$\mathcal{F}_{it}[X] \equiv \int X\,\tilde{\mathrm{p}}_{it}(X)\mathrm{d}X, \quad \textit{with} \quad \tilde{\mathrm{p}}_{it}(X) \propto \exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right) \mathrm{p}(X \mid x_i^t).$$

Proof of Lemma E.1. The first-order-condition for the minimization with respect to m_{it} is given by

$$u(k_{it}, K_t, \xi_t) + \frac{1}{\varpi}\log m_{it} + \frac{1}{\varpi} = 0.$$

Together with the fact that $\mathbb{E}_{it}[m_{it}] = 1$, it follows that

$$m_{it} = \frac{\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)}{\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\right]}.$$

Thus, problem (6.1) can be rewritten as the following problem with risk sensitivity:

$$\max_{k_{it}} -\frac{1}{\varpi}\log\left(\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\right]\right).$$

The first-order-condition for this problem with respect to k_{it} is given by

$$\frac{\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\frac{\partial u(k_{it}, K_t, \xi_t)}{\partial k_{it}}\right]}{\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\right]} = 0.$$

Since

$$\frac{\partial u(k_{it}, K_t, \xi_t)}{\partial k_{it}} = k_{it} - (1-\alpha)\,\xi_t - \alpha K_t,$$

it follows that

$$k_{it} = (1-\alpha)\,\mathbb{E}_{it}\left[\xi_t \frac{\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)}{\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\right]}\right] + \alpha\mathbb{E}_{it}\left[K_t \frac{\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)}{\mathbb{E}_{it}\left[\exp\left(-\varpi u(k_{it}, K_t, \xi_t)\right)\right]}\right].$$

Letting $\frac{\exp(-\varpi u(k_{it}, K_t, \xi_t))}{\mathbb{E}_{it}[\exp(-\varpi u(k_{it}, K_t, \xi_t))]}$ be the Radon-Nikodym derivative completes the proof. □

Proof of Proposition 6.1. Consider the same truncated version of the model used in the proof of Proposition 3.3. From Lemma E.1 we have that the optimal strategy then satisfies that

$$k_i = (1-\alpha)\,\mathcal{F}[\theta \mid x_i] + \alpha\mathcal{F}[K \mid x_i], \tag{E.1}$$

with the distorted posterior given by

$$\tilde{p}(\eta|x_i) \propto \exp\left(-\varpi u(k_i, K, \theta)\right) p(\eta \mid x_i).$$

We proceed with a guess-and-verify strategy. First we guess that

$$k_i = h' B\nu_i + h_0.$$

Substituting this into equation (E.1), it follows that

$$k_i = \left((1-\alpha) A\mathcal{K}' + \alpha h' B\mathcal{K}'\right) \mathcal{F}[\eta \mid B\nu_i] + \alpha h_0.$$

Thus, we need to determine the subjective conditional expectation $\mathcal{F}[\eta \mid B\nu_i]$. We proceed to characterize the distorted posterior $\tilde{p}(\eta \mid B\nu_i)$ by the following three steps:

1. First, the Bayesian posterior $p(\eta \mid B\nu_i)$ is such that

   $$p(\eta \mid B\nu_i) \propto \exp\left(-\frac{1}{2}\left(\eta - \mu_{\eta|B\nu_i}\right)' \Sigma^{-1}_{\eta|B\nu_i}\left(\eta - \mu_{\eta|B\nu_i}\right)\right),$$

   with the conditional mean and variance of given by

   $$\mu_{\eta|B\nu_i} = \mathcal{K}\Omega B' \left(B\Omega B'\right)^{-1} B\nu_i, \quad \text{and} \quad \Sigma_{\eta|B\nu_i} = \mathcal{K}\Omega\mathcal{K}' - \mathcal{K}\Omega B' \left(B\Omega B'\right)^{-1} B\Omega\mathcal{K}'.$$

2. Second, notice that

   $$\begin{aligned} u(k, K, \theta) = & -\frac{1}{2}\left[(1-\alpha)\left(h' B\nu_i + h_0 - A\mathcal{K}'\eta\right)^2 + \alpha\left(h' B\nu_i - h' B\mathcal{K}'\eta\right)^2\right] - \chi A\mathcal{K}'\eta - \frac{1}{2}\gamma\eta'\mathcal{K}A'A\mathcal{K}'\eta \\ &= \text{constant} - \frac{1}{2}\gamma\eta'\mathcal{K}A'A\mathcal{K}'\eta - \frac{1}{2}\left[(1-\alpha)\eta'\mathcal{K}A'A\mathcal{K}'\eta + \alpha\eta'\mathcal{K}B'hh'B\mathcal{K}'\eta\right] \\ &+ \frac{1}{2}\left[(1-\alpha)\left(h_0 + \nu_i' B' h\right) A + \alpha\nu_i' B' hh' B - \chi A\right]\mathcal{K}'\eta \\ &+ \eta'\mathcal{K}\frac{1}{2}\left[(1-\alpha) A'\left(h_0 + h' B\nu_i\right) + \alpha B' hh' B\nu_i - \chi A'\right], \end{aligned}$$

   with the constant independent of η.

3. Finally, putting these results together, the distorted posterior must be such that

   $$\tilde{p}(\eta \mid B\nu_i) \propto \exp\left(-\frac{1}{2}\eta'\tilde{\Sigma}^{-1}_{\eta|B\nu_i}\eta + \frac{1}{2}\tilde{\mu}'_{\eta|B\nu_i}\tilde{\Sigma}^{-1}_{\eta|B\nu_i}\eta + \frac{1}{2}\eta'\tilde{\Sigma}^{-1}_{\eta|B\nu_i}\tilde{\mu}_{\eta|B\nu_i}\right)$$

   where the distorted posterior variance and mean are given by

   $$\tilde{\Sigma}^{-1}_{\eta|B\nu_i} \equiv \Sigma^{-1}_{\eta|B\nu_i} + \mathrm{Q} \quad \text{and} \quad \tilde{\mu}_{\eta|B\nu_i} \equiv \tilde{\Sigma}_{\eta|B\nu_i}\left(\Sigma^{-1}_{\eta|B\nu_i}\mu_{\eta|B\nu_i} + \mathrm{R}B\nu_i\right) + \pi_\mu,$$

with the matrices Q and R and the vector π_μ given by

$$\mathrm{Q} \equiv -\varpi\gamma\mathcal{K}A'A\mathcal{K}' - \varpi\left[(1-\alpha)\mathcal{K}A'A\mathcal{K}' + \alpha\mathcal{K}B'hh'B\mathcal{K}'\right], \tag{E.2}$$

$$\mathrm{R} \equiv -\varpi\mathcal{K}\left[(1-\alpha)A' + \alpha B'h\right]h', \tag{E.3}$$

$$\pi_\mu \equiv -\varpi\left(\Sigma_{\eta|B\nu_i}^{-1} + \mathrm{Q}\right)^{-1}\mathcal{K}\left[(1-\alpha)A'h_0 - \chi A'\right]. \tag{E.4}$$

The distorted expectation under robust preferences can, then, be written as

$$\tilde{\mathbb{E}}\left[\eta \mid B\nu_i\right] = \tilde{\mu}_{\eta|B\nu_i} = \mathrm{M}B\nu_i + \pi_\mu,$$

with the matrix M given by

$$\mathrm{M} \equiv \left(\Sigma_{\eta|B\nu_i}^{-1} + \mathrm{Q}\right)^{-1}\left(\Sigma_{\eta|B\nu_i}^{-1}\mathcal{K}\Omega B'\left(B\Omega B'\right)^{-1} + \mathrm{R}\right). \tag{E.5}$$

Thus, we that that

$$k_i = \left((1-\alpha)A\mathcal{K}' + \alpha h'B\mathcal{K}'\right)\left(\mathrm{M}B\nu_i + \pi_\mu\right) + \alpha h_0.$$

and for the initial guess to be correct the following fixed-point conditions must be satisfied:

$$h' = \left[(1-\alpha)A + \alpha h'B\right]\mathcal{K}'\mathrm{M}, \tag{E.6}$$

$$h_0 = \left[(1-\alpha)A + \alpha h'B\right]\mathcal{K}'\pi_\mu + \alpha h_0. \tag{E.7}$$

In what follows, we first characterize the responsiveness to signals h that solves equation (E.6) and then characterize the bias h_0 that solves equation (E.7).

Characterization of the responsiveness, h. We start by rewriting the equation for the matrix M. Substituting h' from equation (E.6) into equation (E.3), we obtain

$$\mathrm{R} = -\varpi\mathcal{K}\left((1-\alpha)A' + \alpha B'h\right)\left((1-\alpha)A + \alpha h'B\right)\mathcal{K}'\mathrm{M}$$

Plugging this expression for R into the definition of M, equation (E.5), it follows that

$$\left(\Sigma_{\eta|B\nu_i}^{-1} + \mathrm{Q}\right)\mathrm{M} = \left(\Sigma_{\eta|B\nu_i}^{-1}\mathcal{K}\Omega B'\left(B\Omega B'\right)^{-1} - \varpi\mathcal{K}\left((1-\alpha)A' + \alpha B'h\right)\left((1-\alpha)A + \alpha h'B\right)\mathcal{K}'\mathrm{M}\right).$$

Solving for M we get

$$\mathrm{M} = \left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\mathcal{K}\Omega B'\left(B\Omega B'\right)^{-1},$$

where the I_u is the identity matrix of dimension u and the matrix $\tilde{\mathrm{Q}}$ is given by

$$\begin{aligned}\tilde{\mathrm{Q}} &\equiv \mathrm{Q} + \varpi\mathcal{K}\left((1-\alpha)A' + \alpha B'h\right)\left((1-\alpha)A + \alpha h'B\right)\mathcal{K}' \\ &= -\varpi\gamma\mathcal{K}A'A\mathcal{K}' - \varpi\alpha(1-\alpha)\mathcal{K}\left(B'h - A'\right)\left(h'B - A\right)\mathcal{K}'.\end{aligned} \tag{E.8}$$

To ease notation, we define matrices

$$Z_1 \equiv -\varpi\gamma\mathcal{K}A' - \varpi\alpha\left(1-\alpha\right)\mathcal{K}\left(A' - B'h\right), \quad \text{and} \quad Z_2 \equiv -\varpi\alpha\left(1-\alpha\right)\mathcal{K}\left(B'h - A'\right),$$

so that

$$\tilde{Q} = \mathrm{Z}_1 A\mathcal{K}' + \mathrm{Z}_2 h' B\mathcal{K}'.$$

The fixed-point condition (E.6) can, then, be rewritten as

$$h' = \left[\left(1-\alpha\right)A + \alpha h'B\right]\mathcal{K}'\left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\Omega_\eta\mathcal{K}B'\left(B\Omega B'\right)^{-1},$$

where we used the fact that $\mathcal{K}\Omega = \Omega_\eta\mathcal{K}$. Using the Woodbury matrix identity, we obtain

$$\left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\Omega_\eta = \Omega_\eta - \left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\Omega_\eta,$$

so, we can further rewrite the fixed-point condition as

$$\begin{aligned} h' =& \left[\left(1-\alpha\right)A + \alpha h'B\right]\mathcal{K}'\left(\Omega_\eta - \left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\Omega_\eta\right)\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ &= \left[\left(1-\alpha\right)A + \alpha h'B\right]\mathcal{K}'\left(\Omega_\eta - \left(\mathrm{I}_u + \Sigma_{\eta|B\nu_i}\tilde{\mathrm{Q}}\right)^{-1}\Sigma_{\eta|B\nu_i}\left(\mathrm{Z}_1 A\mathcal{K}' + \mathrm{Z}_2 h'B\mathcal{K}'\right)\Omega_\eta\right)\mathcal{K}B'\left(B\Omega B'\right)^{-1} \\ &= \left(1-\alpha+\varkappa_1\right)A\Lambda\Omega B' + \left(\alpha - \varkappa_2\right)h'B\Lambda\Omega B', \end{aligned}$$

where $\Lambda = \mathcal{K}'\mathcal{K}$ and the endogenous scalars $\varkappa_1$ and $\varkappa_2$ are given by

$$\begin{aligned} \varkappa_1 &\equiv -\left[\left(1-\alpha\right)A + \alpha h'B\right]\mathcal{K}'\left(I_u + \Sigma_{\eta|B\nu_i}\tilde{Q}\right)^{-1}\Sigma_{\eta|B\nu_i}Z_1, \\ \varkappa_2 &\equiv \left[\left(1-\alpha\right)A + \alpha h'B\right]\mathcal{K}'\left(I_u + \Sigma_{\eta|B\nu_i}\tilde{Q}\right)^{-1}\Sigma_{\eta|B\nu_i}Z_2. \end{aligned}$$

Solving for h' we obtain

$$h' = \frac{1-\alpha+\varkappa_1}{1-\alpha+\varkappa_2}A\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}, \tag{E.9}$$

where the transformed variance-covariance matrix $\hat{\Omega}$ is given by

$$\hat{\Omega} \equiv \frac{1-\alpha+\varkappa_2}{1-\alpha}\Lambda\Omega + \frac{1}{1-\alpha}\left(\mathrm{I}_m - \Lambda\right)\Omega, \tag{E.10}$$

with I_m denoting the identity matrix of dimension m.

In what follows, we provide expressions for the two endogenous scalars $(\varkappa_1, \varkappa_2)$ such that we can take the limit as $T \to \infty$ and obtain the formulas in Proposition 6.1. For this purpose, it is useful to define

$$\mathrm{X} \equiv \left[\left(1-\alpha\right)A + \alpha h'B\right]\mathcal{K}'\left(\Sigma^{-1}_{\eta|B\nu_i} + \tilde{\mathrm{Q}}\right)^{-1}.$$

Notice that $(\varkappa_1, \varkappa_2)$ can then be written as

$$\varkappa_1 = -\mathrm{X} Z_1 = \varpi\gamma \mathrm{X}\mathcal{K}A' + \varkappa_2, \quad \text{and} \quad \varkappa_2 = -\mathrm{X} Z_2 = \varpi\alpha\,(1-\alpha)\,\mathrm{X}\mathcal{K}\,(A' - B'h)\,.$$

Therefore, it follows that

$$\begin{aligned} \mathrm{X} =&\ [(1-\alpha)\,A + \alpha h' B]\,\mathcal{K}'\left(\Sigma_{\eta|B\nu_i} - \left(\Sigma^{-1}_{\eta|B\nu_i} + \tilde{\mathrm{Q}}\right)^{-1}\tilde{\mathrm{Q}}\Sigma_{\eta|B\nu_i}\right) \\ =&\ [(1-\alpha)\,A + \alpha h' B]\,\mathcal{K}'\Sigma_{\eta|B\nu_i} - \mathrm{X}\tilde{\mathrm{Q}}\Sigma_{\eta|B\nu_i} \\ =&\ [(1-\alpha)\,A + \alpha h' B]\,\mathcal{K}'\Sigma_{\eta|B\nu_i} - \mathrm{X}\,(\mathrm{Z}_1 A\mathcal{K}' + \mathrm{Z}_2 h' B\mathcal{K}')\,\Sigma_{\eta|B\nu_i} \\ =&\ [(1-\alpha)\,A + \alpha h' B]\,\mathcal{K}'\Sigma_{\eta|B\nu_i} + (\varkappa_1 A\mathcal{K}' - \varkappa_2 h' B\mathcal{K}')\,\Sigma_{\eta|B\nu_i} \\ =&\ (1-\alpha+\varkappa_1)\,A\mathcal{K}'\Sigma_{\eta|B\nu_i} + (\alpha - \varkappa_2)\,h' B\mathcal{K}'\Sigma_{\eta|B\nu_i}. \end{aligned}$$

Thus, since $\varkappa_1 - \varkappa_2 = \varpi\gamma\mathrm{X}\mathcal{K}A'$, we have that,

$$\varkappa_1 - \varkappa_2 = \varpi\gamma\,(1-\alpha+\varkappa_1)\,A\mathcal{K}'\Sigma_{\eta|B\nu_i}\mathcal{K}A' + \varpi\gamma\,(\alpha-\varkappa_2)\,h' B\mathcal{K}'\Sigma_{\eta|B\nu_i}\mathcal{K}A'. \tag{E.11}$$

Next, notice that

$$\mathrm{X} = \mathrm{X}\Sigma^{-1}_{\eta|B\nu_i}\Sigma_{\eta|B\nu_i} = \mathrm{X}\Sigma^{-1}_{\eta|B\nu_i}\mathcal{K}\Omega\mathcal{K}' - \mathrm{X}\Sigma^{-1}_{\eta|B\nu_i}\mathcal{K}\Omega B'\,(B\Omega B')^{-1}\,B\Omega\mathcal{K}' = \mathrm{X}\Sigma^{-1}_{\eta|B\nu_i}\mathcal{K}\Omega\mathcal{K}' - h' B\Omega\mathcal{K}',$$

where the second equality uses the definition of $\Sigma_{\eta|B\nu_i}$ and the last equality uses the fact that

$$h' = \mathrm{X}\Sigma^{-1}_{\eta|B\nu_i}\mathcal{K}\Omega B'\,(B\Omega B')^{-1}\,.$$

Rearranging terms and right-multiplying $(\mathcal{K}\Omega\mathcal{K}')^{-1}\,\mathcal{K}\Omega B'$ to both sides of the equation, we obtain

$$\mathrm{X}\Sigma^{-1}_{\eta|B\nu_i}\mathcal{K}\Omega B' = \mathrm{X}\,(\mathcal{K}\Omega\mathcal{K}')^{-1}\,\mathcal{K}\Omega B' + h' B\Omega\mathcal{K}'\,(\mathcal{K}\Omega\mathcal{K}')^{-1}\,\mathcal{K}\Omega B' = \mathrm{X}\mathcal{K}B' + h' B\Lambda\Omega B'.$$

Further, since $\mathrm{X}\Sigma^{-1}_{\eta|B\nu_i}\mathcal{K}\Omega B' = h' B\Omega B'$, it follows that

$$\mathrm{X}\mathcal{K}B'h = h' B\,(\mathrm{I}_m - \Lambda)\,\Omega B'h.$$

Hence, we have that

$$\begin{aligned} \varkappa_2 &= \varpi\alpha\,(1-\alpha)\,\mathrm{X}\mathcal{K}A' - \varpi\alpha\,(1-\alpha)\,\mathrm{X}\mathcal{K}B'h && \text{(E.12)} \\ &= \frac{\alpha\,(1-\alpha)}{\gamma}\,(\varkappa_1 - \varkappa_2) - \varpi\alpha\,(1-\alpha)\,h' B\,(\mathrm{I}_m - \Lambda)\,\Omega B'h, && \text{(E.13)} \end{aligned}$$

where we use the fact that $\varpi\gamma\mathrm{X}\mathcal{K}A' = \varkappa_1 - \varkappa_2$.

Given the above results, we are left with taking the limit as $T \to \infty$ of the truncated problem. In particular, we have that

$$\lim_{T\to\infty} A\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} = p\left(L;w,\alpha\right) \qquad \lim_{T\to\infty} A\mathcal{K}'\Sigma_{\eta|B\nu_i}\mathcal{K}A' = \mathbb{V}_{it}\left(\xi_t\right)$$
$$\lim_{T\to\infty} h'B\mathcal{K}'\Sigma_{\eta|B\nu_i}\mathcal{K}A' = \mathbb{COV}_{it}\left(K_t,\xi_t\right) \qquad \lim_{T\to\infty} h'B\left(\mathrm{I}_m-\Lambda\right)\Omega B'h = \mathbb{DISP}\left(k_{it}\right)$$

which, together with equations (E.9), (E.10), (E.11), and (E.13), completes the characterization of the responsiveness to signals.

Characterization of the bias, h_0. From the fixed-point condition (E.7) and the definition of π_μ in equation (E.4), it follows that

$$(1-\alpha)\,h_0 = \varpi\left[(1-\alpha)\,A+\alpha h'B\right]\mathcal{K}'\left(\Sigma^{-1}_{\eta|B\nu_i}+\mathrm{Q}\right)^{-1}\mathcal{K}\left[\chi A'-(1-\alpha)\,A'h_0\right],$$

which can be solved for h_0 implying

$$h_0 = \frac{\chi\varpi\mathrm{Y}}{(1-\alpha)\,(1+\varpi\mathrm{Y})},$$

with Y given by

$$\mathrm{Y} \equiv \left[(1-\alpha)\,A+\alpha h'B\right]\mathcal{K}'\left(\Sigma^{-1}_{\eta|B\nu_i}+\mathrm{Q}\right)^{-1}\mathcal{K}A'.$$

Using the definition of $\tilde{\mathrm{Q}}$ in equation (E.8) and the Woodbury matrix identity, it follows that

$$\begin{aligned}
\left(\Sigma^{-1}_{\eta|B\nu_i}+\mathrm{Q}\right)^{-1} &= \left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}-\varpi\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)\left((1-\alpha)\,A+\alpha h'B\right)\mathcal{K}'\right)^{-1}\\
&= \left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}\right)^{-1}+\\
&\quad \frac{\varpi\left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}\right)^{-1}\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)\left((1-\alpha)\,A+\alpha h'B\right)\mathcal{K}'\left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}\right)^{-1}}{1-\varpi\left((1-\alpha)\,A+\alpha h'B\right)\mathcal{K}'\left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}\right)^{-1}\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)}\\
&= \left(\Sigma^{-1}_{\eta|B\nu_i}+\tilde{\mathrm{Q}}\right)^{-1}+\frac{\varpi\mathrm{X}'\mathrm{X}}{1-\varpi\mathrm{X}\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Y} &= \mathrm{X}\mathcal{K}A'+\frac{\varpi\left[(1-\alpha)\,A+\alpha h'B\right]\mathcal{K}'\mathrm{X}'\mathrm{X}\mathcal{K}A'}{1-\varpi\mathrm{X}\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)}\\
&= \frac{\mathrm{X}\mathcal{K}A'}{1-\varpi\mathrm{X}\mathcal{K}\left((1-\alpha)\,A'+\alpha B'h\right)}\\
&= \frac{\frac{\varkappa_1-\varkappa_2}{\varpi\gamma}}{1-\left(\frac{1-\alpha}{\gamma}\right)(\varkappa_1-\varkappa_2)-\varpi\alpha h'B\left(\mathrm{I}_m-\Lambda\right)\Omega B'h},
\end{aligned}$$

where the last equality uses the fact that

$$\varkappa_1-\varkappa_2 = \varpi\gamma\mathrm{X}\mathcal{K}A', \quad \text{and} \quad \mathrm{X}\mathcal{K}B'h = h'B\left(\mathrm{I}_m-\Lambda\right)\Omega B'h.$$

Therefore, we have that

$$h_0 = \frac{\chi(\varkappa_1 - \varkappa_2)}{(1-\alpha)\left(\gamma + \alpha\left(\varkappa_1 - \varkappa_2\right) - \gamma\varpi\alpha h' B\left(\mathrm{I}_m - \Lambda\right)\Omega B' h\right)}.$$

Finally, taking the limit as $T \to \infty$ leads to

$$\mathcal{B} = \lim_{T\to\infty} h_0 = \frac{\chi(\varkappa_1 - \varkappa_2)}{(1-\alpha)\left(\gamma + \alpha\left(\varkappa_1 - \varkappa_2\right) - \gamma\varpi\alpha\mathbb{DISP}\left(k_{it}\right)\right)}.$$

□

Proof of Corollary 6.1. Observe that, by using (3.16), the expression of w under smooth model (3.15) can be transformed into

$$w = \left[\frac{1}{\tau_\mu} - \lambda\left(1-\alpha\right)\left(\mathbb{V}\left(\xi_t - K_t\right) + r\frac{1+w}{w}\left(1-\mathcal{S}\right)\mathbb{V}\left(\xi_t\right)\right)\right]^{-1} \tag{E.14}$$

Take any pair (w, r) and the associated sensitivity $\mathcal{S}$ that would arise from robust preferences. We may solve $\left(\lambda, \sigma_\mu^2\right)$ from (3.16) and (E.14). Note that the first condition $w \geq 0, r \geq 0, \mathcal{S} \leq 1$ ensures that Assumption 2 can be satisfied and the second condition$(1-\mathcal{S})\left(\frac{\gamma w}{(1+w)r} - \frac{(1-\alpha)(1+w)r}{w}\right) + \gamma > (1-\alpha)\frac{\mathbb{V}(\xi_t - K_t)}{\mathbb{V}(\xi_t)}$ ensures that the resulted $\tau_\mu > 0$.

□

F Value of Information

In this appendix, we demonstrate that the value of information increases with the amount of ambiguity. To start with, as a simplification, we restrict our attention to the situation where the idiosyncratic noises share a common variance σ_ϵ^2. Specifically, we investigate the sign of the following cross-derivative for agent i:

$$\mathrm{D} \equiv -\frac{\mathrm{d}^2 \mathrm{V}\left(\sigma_\epsilon^2; \bar{g}\left(x_{-i}^t\right)\right)}{\mathrm{d}\sigma_\epsilon^2 \mathrm{d}\tau_\mu},$$

where

$$\mathrm{V}\left(\sigma_\epsilon^2; \bar{g}\left(x_{-i}^t\right)\right) \equiv \phi^{-1}\left(\int_{\mu^t} \phi\left(\mathbb{E}^{\mu^t}\left[u\left(k_{it}, K_t, \xi_t\right)\right]\right) p\left(\mu^t\right) \mathrm{d}\mu^t\right),$$

and $\bar{g}\left(x_{-i}^t\right)$ denotes the strategies taken by all other agents. The derivative $-\mathrm{dV}\left(\sigma_\epsilon^2; \bar{g}\left(x_{-i}^t\right)\right)/\mathrm{d}\sigma_\epsilon^2$ captures the effect on the agent's objective function of an increase in signal precision, thereby quantifying the value of extra information. As a result, a positive sign of the cross-derivative D reflects that a higher level of ambiguity increases the value of information.

We allow D to depend on the strategies of the other agents $\bar{g}\left(x_{-i}^t\right)$. This approach focuses our analysis on the value of information from the perspective of agent i, without imposing a symmetric equilibrium a priori. As a result, this notion of the value of information is ready to be incorporated into a rational inattention framework with some information acquisition cost function. This way of measuring the value of information is also consistent with our framework of persistent learning, where all private information shares the same precision so that a marginal change in σ_ϵ^2 changes the precision of all private information. In a generic environment where the precision of different sources of private information can differ substantially, our notion of the value of information can be equivalently understood as increasing the precision of all private information by the same amount.

In what follows, through the lens of a set of lemmas, we demonstrate that $\mathrm{D} > 0$, i.e., the value of information increases with the amount of ambiguity. We begin with Lemma F.1, which analytically characterizes the value of information.

Lemma F.1. *If $\phi(\cdot)$ takes the CAAA form, i.e., $\phi(x) = -\frac{1}{\lambda}\exp(-\lambda x)$, the value of information equals the equilibrium cross-sectional dispersion of actions:*

$$-\frac{\mathrm{dV}(\sigma_\epsilon^2; \bar{g}(x_{-i}^t))}{\mathrm{d}\sigma_\epsilon^2} = \frac{1}{2\sigma_\epsilon^2}\,\mathbb{E}[(k_{it} - K_t)^2].$$

Proof. We start the proof with the truncated economy as in the proof of Proposition 3.3. As a result, the strategies of individual agent i and of the other agents are respectively given by

$$k_i = h' B \nu_i + h_0, \quad \text{and} \quad K = \bar{h}' B \Lambda \nu_i + \bar{h}_0.$$

When $\phi(\cdot)$ takes the CAAA form, the ex-ante value of agent i is such that

$$\mathrm{V}(\sigma_\epsilon^2; \bar{h}, h_0) \equiv -\frac{1}{\lambda} \ln \left(\int_\mu \exp\left(-\lambda \mathbb{E}^\mu[u(k_i, K, \theta)]\right) p(\mu) \mathrm{d}\mu \right)$$
$$\text{s.t.} \quad k_i = h' B \nu_i + h_0 \quad \text{and} \quad K = \bar{h}' B \Lambda \nu_i + h_0.$$

Taking derivative with respect to σ_ϵ^2 leads to

$$\begin{aligned}
\frac{\mathrm{dV}(\sigma_\epsilon^2; \bar{h}, h_0)}{\mathrm{d}\sigma_\epsilon^2} &= \frac{\int_\mu \exp\left(-\lambda \mathbb{E}^\mu[u(k_i, K, \theta)]\right) \left(\frac{\partial \mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial h} \frac{\mathrm{d}h}{\mathrm{d}\sigma_\epsilon^2} + \frac{\partial \mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial h_0} \frac{\mathrm{d}h_0}{\mathrm{d}\sigma_\epsilon^2} + \frac{\partial \mathbb{E}^\mu[u(k_i,K,\theta)]}{\partial \sigma_\epsilon^2} \right) p(\mu) \mathrm{d}\mu}{\int_\mu \exp\left(-\lambda \mathbb{E}^\mu[u(k_i, K, \theta)]\right) p(\mu) \mathrm{d}\mu} \\
&= \int_\mu \frac{\partial \mathbb{E}^\mu[u(k_i, K, \theta)]}{\partial \sigma_\epsilon^2} \hat{p}(\mu) \mathrm{d}\mu + \frac{\mathrm{d}h}{\mathrm{d}\sigma_\epsilon^2} \int_\mu \frac{\partial \mathbb{E}^\mu[u(k_i, K, \theta)]}{\partial h} \hat{p}(\mu) \mathrm{d}\mu + \frac{\mathrm{d}h_0}{\mathrm{d}\sigma_\epsilon^2} \int_\mu \frac{\partial \mathbb{E}^\mu[u(k_i, K, \theta)]}{\partial h_0} \hat{p}(\mu) \mathrm{d}\mu,
\end{aligned}$$

where $\hat{p}(\mu)$ is the (ex-ante) distorted subjective belief given by

$$\hat{p}(\mu) \propto \exp\left(-\lambda \mathbb{E}^\mu[u(k_i, K, \theta)]\right) p(\mu).$$

Note that the first-order conditions that pin down the optimal sensitivity h and bias h_0 are such that

$$\int_\mu \frac{\partial \mathbb{E}^\mu[u(k_i, K, \theta)]}{\partial h} \hat{p}(\mu) \mathrm{d}\mu = \int_\mu \frac{\partial \mathbb{E}^\mu[u(k_i, K, \theta)]}{\partial h_0} \hat{p}(\mu) \mathrm{d}\mu = 0.$$

Denote $\mathcal{K}$ and $\mathcal{G}$ by

$$\mathcal{K} \equiv \left[\mathrm{I}_u, 0_{u,m-u}\right], \quad \text{and} \quad \mathcal{G} \equiv \left[0_{m-u,u}, \mathrm{I}_{m-u}\right].$$

It can then be shown that

$$\begin{aligned}
\mathbb{E}^\mu[u(k_i, K, \theta)] =& -\frac{1}{2}(1-\alpha)\mathbb{E}^\mu[(h'B(\mathcal{K}'\eta + \mathcal{G}'\epsilon_i) + h_0 - a'\eta)^2] \\
& - \frac{1}{2}\alpha \mathbb{E}^\mu[(h'B(\mathcal{K}'\eta + \mathcal{G}'\epsilon_i) + h_0 - \bar{h}'B\mathcal{K}'\eta - \bar{h}_0)^2] - \mathbb{E}^\mu[\chi a'\eta + \frac{1}{2}\gamma a'\eta\eta' a] \\
=& -\frac{1}{2} h' B (I - \Lambda) B' h \, \sigma_\epsilon^2 + \mathbb{Z}(\mu, \sigma_\eta^2, h, h_0, \bar{h}, \bar{h}_0),
\end{aligned}$$

where $\Lambda = \mathcal{K}'\mathcal{K}$, and $\mathbb{Z}(\mu, \sigma_\eta^2, h, h_0, \bar{h}, \bar{h}_0)$ are independent of σ_ϵ. Therefore, we have

$$-\frac{\mathrm{dV}(\sigma_\epsilon^2; \bar{h}, h_0)}{\mathrm{d}\sigma_\epsilon^2} = -\int_\mu \frac{\partial \mathbb{E}^\mu[u(k_i, K, \theta)]}{\partial \sigma_\epsilon^2} \hat{p}(\mu) \mathrm{d}\mu = \frac{1}{2} h' B (I - \Lambda) B' h = \frac{1}{2\sigma_\epsilon^2} h' B (I - \Lambda) \Omega B' h.$$

Taking the limit as $T \to \infty$ of the truncated problem yields

$$\lim_{T \to +\infty} -\frac{\mathrm{dV}(\sigma_\epsilon^2; \bar{h}, h_0)}{\mathrm{d}\sigma_\epsilon^2} = -\frac{\mathrm{dV}(\sigma_\epsilon^2; \bar{g}(x_{-i}^t))}{\mathrm{d}\sigma_\epsilon^2}, \quad \text{and} \quad \lim_{T \to +\infty} h' B (I - \Lambda) \Omega B' h = \mathbb{E}[(k_{it} - K_t)^2].$$

Thus, the value of information equals the equilibrium cross-sectional dispersion of actions. □

Does higher ambiguity increase the value of information? Providing an answer to this question is equivalent to analyzing whether the cross-sectional dispersion of actions increases with the amount of ambiguity τ_μ. Our equivalence result suggests that τ_μ shapes cross-sectional dispersion by affecting the two endogenous scalars w and r. In what follows, we first characterize how w and r affect the cross-sectional dispersion of actions (Lemma F.2). Intuitively, increases in either w or r should increase the cross-sectional dispersion, given that both higher w and r contribute to more overreactions. Lemma F.2 confirms this intuition.

Lemma F.2. *The cross-sectional dispersion of actions is increasing in both w and r:*

$$\frac{\partial \mathbb{E}\left[(k_{it}-K_t)^2\right]}{\partial w} > 0, \quad \textit{and} \quad \frac{\partial \mathbb{E}\left[(k_{it}-K_t)^2\right]}{\partial r} > 0.$$

Proof. Again, we start the proof with the truncated economy, in which $h(w,r)'B(I-\Lambda)\Omega B'h(w,r)$ denotes cross-sectional dispersion. Further, denote $\hat{h}'(w)$ as the truncated version of $p(L;w,\alpha)$, namely the forecasting rule of the (w,α)-modified signal process in Section 3.3. Then, we have that

$$h'(w,r) = (1+r)\hat{h}'(w),$$

which implies that

$$h(w,r)'B(I-\Lambda)\Omega B'h(w,r) = (1+r)^2\hat{h}(w)'B(I-\Lambda)\Omega B'\hat{h}(w).$$

It is then straightforward to see that

$$\frac{\partial h(w,r)'B(I-\Lambda)\Omega B'h(w,r)}{\partial r} > 0.$$

In what follows, we proceed to prove that $\hat{h}(w)'B(I-\Lambda)\Omega B'\hat{h}(w)$ is increasing in w. Utilizing our equivalence results, it can be shown that

$$\begin{aligned}
\hat{h}'(w) =& A\left((1+w)\Lambda\Omega + (1-\alpha)^{-1}(I-\Lambda)\Omega\right)B'\left(B\left((1+w)\Lambda\Omega + (1-\alpha)^{-1}(I-\Lambda)\Omega\right)B'\right)^{-1} \\
=& A\left(\Lambda\Omega + (1+w)^{-1}(1-\alpha)^{-1}(I-\Lambda)\Omega\right)B'\left(B\left(\Lambda\Omega + (1+w)^{-1}(1-\alpha)^{-1}(I-\Lambda)\Omega\right)B'\right)^{-1} \\
=& A\Omega B'\left(B\left(\Lambda\Omega + (1+w)^{-1}(I-\Lambda)\Omega\right)B'\right)^{-1} \\
=& A\Omega B'\left(B\left(\Lambda\Omega_\alpha + (1+w)^{-1}(I-\Lambda)\Omega_\alpha\right)B'\right)^{-1},
\end{aligned}$$

where $\Omega_\alpha \equiv \Lambda\Omega + (1-\alpha)^{-1}(I-\Lambda)\Omega$. As a result, taking the derivative with respect to w leads to

$$\frac{\mathrm{d}\hat{h}'(w)}{\mathrm{d}w} = (1+w)^{-2}\,\hat{h}'(w)B(I-\Lambda)\Omega_\alpha B'\left(B(\Lambda\Omega_\alpha + (1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}.$$

Therefore, we have that

$$
\begin{aligned}
&(1+w)^2\frac{\mathrm{d}\hat{h}(w)'B(I-\Lambda)\Omega B'\hat{h}(w)}{\mathrm{d}w}\\
&=\hat{h}'(w)B(I-\Lambda)\Omega_\alpha B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}B(I-\Lambda)\Omega B'\hat{h}(w)\\
&\quad+\hat{h}(w)'B(I-\Lambda)\Omega B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}B(I-\Lambda)\Omega_\alpha B'\hat{h}(w)\\
&=(1-\alpha)^{-1}\hat{h}'(w)B(I-\Lambda)\Omega B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}B(I-\Lambda)\Omega B'\hat{h}(w)\\
&\quad+(1-\alpha)^{-1}\hat{h}(w)'B(I-\Lambda)\Omega B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}B(I-\Lambda)\Omega B'\hat{h}(w)\\
&=2(1-\alpha)^{-1}\varpi\Pi^{-1}\varpi',
\end{aligned}
$$

where $\varpi\equiv\hat{h}'(w)B(I-\Lambda)\Omega B'$ and $\Pi\equiv B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'$. Notice that the matrix Π^{-1} is symmetric and positive semi-definite, hence so is Π. We then conclude that

$$\frac{\mathrm{d}\hat{h}(w)'B(I-\Lambda)\Omega B'\hat{h}(w)}{\mathrm{d}w}>0\quad\Leftrightarrow\quad\frac{\partial h(w,r)'B(I-\Lambda)\Omega B'h(w,r)}{\partial w}>0.$$

Finally, taking the limit as $T\to\infty$ of the truncated problem results in

$$\frac{\mathrm{d}\mathbb{E}[(k_{it}-K_t)^2]}{\mathrm{d}r}=\lim_{T\to\infty}\frac{\partial h(w,r)'B(I-\Lambda)\Omega B'h(w,r)}{\partial r}>0,$$

and

$$\frac{\mathrm{d}\mathbb{E}[(k_{it}-K_t)^2]}{\mathrm{d}w}=\lim_{T\to\infty}\frac{\partial h(w,r)'B(I-\Lambda)\Omega B'h(w,r)}{\partial w}>0.$$

□

In the last step, we analyze how changes in τ_μ affect w and r directly. To enjoy an analytical result, we abstract out r by setting $\gamma=0$.

Lemma F.3. *The endogenous scaler w is increasing in τ_μ if $\gamma=0$.*

Proof. When $\gamma=0$, it can be shown that

$$w=\frac{1}{\frac{1}{\tau_\mu}-\lambda(1-\alpha)(A-h'B)\Lambda\Omega(A-h'B)'}. \tag{F.1}$$

Similar to the proof of Lemma F.2 and using the same notation, it can be shown that

$$h=A\Omega B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1},$$

which implies that

$$\frac{\mathrm{d}h}{\mathrm{d}w}=(1+w)^{-2}h'B(I-\Lambda)\Omega_\alpha B'\left(B(\Lambda\Omega_\alpha+(1+w)^{-1}(I-\Lambda)\Omega_\alpha)B'\right)^{-1}.$$

Therefore, we can show that

$$
\begin{aligned}
&\frac{\mathrm{d}(A-h'B)\Lambda\Omega(A-h'B)'}{\mathrm{d}w} \\
&= -2(1+w)^{-1}(A-h'B)\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B'h \\
&= 2(1+w)^{-1}(h'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B'h - h'B(I-\Lambda)\hat{\Omega}B'h) \\
&= 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B' - B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B\Lambda\hat{\Omega}B' - B(I-\Lambda)\hat{\Omega}B')h \\
&= 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B' - B(I-\Lambda)\hat{\Omega}B')h \\
&= 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} - I)B(I-\Lambda)\hat{\Omega}B'h \\
&= -2(1+w)^{-1}(h'B(I-\Lambda)\hat{\Omega}B')\left(B\hat{\Omega}B'\right)^{-1}(B(I-\Lambda)\hat{\Omega}B'h) < 0,
\end{aligned}
$$

where we denote $\hat{\Omega} = \Lambda\Omega_\alpha + (1+w)^{-1}(I-\Lambda)\Omega_\alpha$. It can be further shown that

$$
\begin{aligned}
\frac{\mathrm{d}(A-h'B)\Lambda\Omega(A-h'B)'}{\mathrm{d}w} &= 2(1+w)^{-1}(h'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B'h \\
&= 2(1+w)^{-1}(h'B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B'h - h'B(I-\Lambda)\hat{\Omega}B'h) \\
&= 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B' - B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B\Lambda\hat{\Omega}B' - B(I-\Lambda)\hat{\Omega}B')h \\
&= 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1}B(I-\Lambda)\hat{\Omega}B' - B(I-\Lambda)\hat{\Omega}B')h \\
&= 2(1+w)^{-1}h'(B\Lambda\hat{\Omega}B'\left(B\hat{\Omega}B'\right)^{-1} - I)B(I-\Lambda)\hat{\Omega}B'h \\
&= -2(1+w)^{-1}(h'B(I-\Lambda)\hat{\Omega}B')\left(B\hat{\Omega}B'\right)^{-1}(B(I-\Lambda)\hat{\Omega}B'h) < 0.
\end{aligned}
$$

Denote the right-hand side of equation (F.1) by $\mathrm{RHS}(\tau_\mu, w)$ and the left-hand side by $\mathrm{LHS}(w)$. It is then straightforward to demonstrate that

$$
\frac{\mathrm{dLHS}(w)}{\mathrm{d}w} > 0, \quad \frac{\partial \mathrm{RHS}(\tau_\mu, w)}{\partial w} < 0, \quad \text{and} \quad \frac{\partial \mathrm{RHS}(\tau_\mu, w)}{\partial \tau_\mu} < 0,
$$

which jointly proves that

$$
\frac{\mathrm{d}w}{\mathrm{d}\tau_\mu} > 0.
$$

□

Lemma F.1, Lemma F.3, and Lemma F.2 combined establish the desired result, that the value of information increases with the amount of ambiguity if $\gamma = 0$:

$$
\mathrm{D} > 0 \quad \text{if} \quad \gamma = 0.
$$

In the general case where $\gamma > 0$, proving that $\mathrm{D} > 0$ turns out to be challenging. However, extensive numerical exercises suggest that the value of information continues to increase with the level of ambiguity in this more complex scenario. Intuitively, with $\gamma > 0$, there is an additional channel of overreaction, namely, the scalar $r > 0$, which leads to a higher utilization of information. It is the intricate interaction between w and r, however, that complicates the analytical analysis.

G Ambiguity about Variance

In this section, we explore the cases in which there is ambiguity about the variance of the fundamental and about the variance of the noise, respectively.

G.1 Ambiguity about the variance of the fundamental

We start with the case that agents perceive ambiguity about the variance of the fundamental. Specifically, we assume that the fundamental ξ follows a normal distribution with mean 0 and variance $\sigma^2_{\xi,*}$: $\xi \sim \mathcal{N}\left(0, \sigma^2_{\xi,*}\right)$. Agents exhibit ambiguity regarding the true variance of the fundamental, $\sigma^2_{\xi,*}$. We let Γ_ξ be the the range of possible values for the variance of the fundamental, σ^2_ξ. Analysts believe that $\sigma^2_\xi \in \Gamma_\xi$ and have some prior belief about Γ_ξ with density distribution given by $p\left(\sigma^2_\xi\right)$. To ensure that strategies based on Bayesian inference and ambiguity neutrality coincide, we impose the following assumption on the agents' prior belief:

Assumption 3. *The prior belief of the agent is such that*

$$\int_{\Gamma_\xi} \sigma^2_\xi \, p\left(\sigma^2_\xi\right) \mathrm{d}\sigma^2_\xi = \sigma^2_{\xi,*} \, .$$

Similar to the setup of ambiguity about the mean of the fundamental, each agent receives a private signal

$$x_i = \xi + \varepsilon_i, \quad \text{with} \quad \varepsilon_i \sim \mathcal{N}\left(0, \sigma^2_\epsilon\right).$$

Agents are ambiguity-averse and select a strategy $g(x_i)$ to minimize the following objective:

$$\mathcal{L}(g) = \phi^{-1}\left(\int_{\Gamma_\xi} \phi\left(\mathbb{E}^{\sigma^2_\xi}\left[(g(x_i)-\xi)^2 - \chi\xi\right]\right) p(\sigma^2_\xi)\mathrm{d}\sigma^2_\xi\right),$$

where $\phi(x) = \frac{1}{\lambda}\exp(\lambda x)$ takes the CAAA form with λ representing the degree of ambiguity aversion. Finally, we restrict our analysis to linear strategies such that

$$g(x_i) = s x_i + b, \tag{G.1}$$

which facilitates a direct comparison with our baseline setup, where ambiguity pertains to the mean of the fundamental.

The following proposition suggests that ambiguity has a more limited effect, leading to an optimal linear strategy that exhibits higher sensitivity compared to the rational RE benchmark, but no bias.

Proposition G.1. *When agents are ambiguity-averse, $\lambda > 0$, the optimal linear strategy exhibits higher sensitivity than the RE benchmark and features no bias:*

$$s^* > s^{RE} \equiv \frac{\sigma^2_{\xi,*}}{\sigma^2_{\xi,*} + \sigma^2_\epsilon}, \quad \textit{and} \quad b^* = 0.$$

Proof. Given the restriction to linear strategies, the objective function of the agents can be written as a function of the sensitivity, s, and bias, b, as follows

$$\mathcal{L}(s,b) = \frac{1}{\lambda}\ln\left(\int_{\Gamma_\xi} \exp\left(\lambda\left((s-1)^2\sigma_\xi^2 + s^2\sigma_\epsilon^2\right)\right)p\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2\right) + \frac{1}{2}b^2.$$

The zero-bias result is straight-forward: the FOC with respect to bias b is such that

$$\frac{\partial\mathcal{L}(s,b)}{\partial b} = b = 0.$$

To characterize the optimal of sensitivity, s, we consider the corresponding FOC,

$$\frac{\partial\mathcal{L}(s,b)}{\partial s} = \frac{\int_{\Gamma_\xi}\exp\left(\lambda\left((s-1)^2\sigma_\xi^2 + s^2\sigma_\epsilon^2\right)\right)\left[(s-1)\sigma_\xi^2 + s\sigma_\varepsilon^2\right]p\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2}{\int_{\Gamma_\xi}\exp\left(\lambda\left((s-1)^2\sigma_\xi^2 + s^2\sigma_\epsilon^2\right)\right)p\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2} = 0,$$

which is equivalent to

$$s\,\sigma_\varepsilon^2 = (1-s)\int_{\Gamma_\xi}\sigma_\xi^2\,\hat{p}\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2,$$

where the distorted belief $\hat{p}(\sigma_\xi^2)$ is such that

$$\hat{p}(\tau_\xi) \propto \exp\left(\lambda(s-1)^2\sigma_\xi^2\right)p\left(\sigma_\xi^2\right).$$

Notice that, relative to the agents' prior $p(\sigma_\xi^2)$, the distorted belief $\hat{p}(\sigma_\xi^2)$ puts higher weights on the larger σ_ξ^2 in Γ_ξ: $\hat{p}(\sigma_\xi^2)$ first-order stochastically dominates $p(\sigma_\xi^2)$. It follows that

$$\int_{\Gamma_\xi}\sigma_\xi^2\,\hat{p}\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2 \geq \int_{\Gamma_\xi}\sigma_\xi^2\,p\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2 = \sigma_{\xi,*}^2\,,$$

and, therefore,

$$s^* = \frac{\int_{\Gamma_\xi}\sigma_\xi^2\,\hat{p}\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2}{\int_{\Gamma_\xi}\sigma_\xi^2\,\hat{p}\left(\sigma_\xi^2\right)\mathrm{d}\sigma_\xi^2 + \sigma_\epsilon^2} > \frac{\sigma_{\xi,*}^2}{\sigma_{\xi,*}^2 + \sigma_\epsilon^2} = s^{\mathrm{RE}}.$$

□

G.2 Ambiguity about the variance of signal noise

We proceed to analyze the effect of ambiguity about the variance of the noise instead. Similar to the setup of Section G.1, we assume that the fundamental ξ follows a normal distribution with mean 0 and variance σ_ξ^2, $\xi \sim \mathcal{N}\left(0,\sigma_\xi^2\right)$. Moreover, each agent receives a private signal

$$x_i = \xi + \varepsilon_i, \quad \text{with} \quad \varepsilon_i \sim \mathcal{N}\left(0,\sigma_{\epsilon,*}^2\right).$$

Agents face ambiguity regarding the true variance of the noise, denoted as $\sigma^2_{\epsilon,*}$. We let Γ_ϵ represent the range of possible values for this variance. Agents maintain a belief that σ^2_ϵ lies within Γ_ϵ and hold a prior distribution over this range, represented by $p(\sigma^2_\epsilon)$. To ensure that strategies based on Bayesian inference and ambiguity neutrality coincide, we impose the following assumption on the agents' prior belief:

Assumption 4. *The prior belief of the agent is such that*

$$\int_{\Gamma_\epsilon} \sigma^2_\epsilon \, p(\sigma^2_\epsilon) \mathrm{d}\sigma^2_\epsilon = \sigma^2_{\epsilon,*} \, .$$

Agents are ambiguity averse and select a strategy $g(x_i)$ to minimize the following objective:

$$\mathcal{L}(g) = \phi^{-1}\left(\int_{\Gamma_\epsilon} \phi\left(\mathbb{E}^{\sigma^2_\epsilon}[(g(x_i)-\xi)^2 - \chi\xi]\right) p(\sigma^2_\epsilon)\mathrm{d}\sigma^2_\epsilon\right),$$

where $\phi(x) = \frac{1}{\lambda}\exp(\lambda x)$ takes the CAAA form with λ representing the degree of ambiguity aversion. Finally, we restrict our analysis to linear strategies as in equation (G.1).

The following proposition states that ambiguity has not only a more limited effect but an opposite one on sensitivity when ambiguity is on the variance of noise: the optimal linear strategy exhibits lower sensitivity compared to the rational RE benchmark, while featuring no bias.

Proposition G.2. *When agents are ambiguity averse, $\lambda > 0$, the optimal linear strategy exhibits higher sensitivity than the RE benchmark and features no bias:*

$$s^* < s^{RE} \equiv \frac{\sigma^2_\xi}{\sigma^2_\xi + \sigma^2_{\epsilon *}}, \quad \textit{and} \quad b^* = 0.$$

Proof. Given the restriction to linear strategies, the objective function of the agents can be written as a function of the sensitivity, s, and bias, b, as follows:

$$\mathcal{L}(s,b) = \frac{1}{\lambda}\ln\left(\int_{\Gamma_\epsilon} \exp\left(\lambda\left((s-1)^2\sigma^2_\xi + s^2\sigma^2_\epsilon\right)\right) p(\sigma^2_\epsilon)\mathrm{d}\sigma^2_\epsilon\right) + \frac{1}{2}b^2.$$

The zero-bias result is straightforward: the first-order condition with respect to bias b is such that

$$\frac{\partial\mathcal{L}(s,b)}{\partial b} = b = 0.$$

To characterize the optimal sensitivity, s, we consider the corresponding first-order condition,

$$\frac{\partial\mathcal{L}(s,b)}{\partial s} = \frac{\int_{\Gamma_\epsilon} \exp\left(\lambda\left((s-1)^2\sigma^2_\xi + s^2\sigma^2_\epsilon\right)\right)\left[(s-1)\sigma^2_\xi + s\sigma^2_\varepsilon\right] p(\sigma^2_\epsilon)\mathrm{d}\sigma^2_\epsilon}{\int_{\Gamma_\epsilon} \exp\left(\lambda\left((s-1)^2\sigma^2_\xi + s^2\sigma^2_\epsilon\right)\right) p(\sigma^2_\epsilon)\mathrm{d}\sigma^2_\epsilon} = 0,$$

which is equivalent to

$$s\int_{\Gamma_\epsilon} \sigma^2_\varepsilon \, \hat{p}(\sigma^2_\epsilon)\mathrm{d}\sigma^2_\epsilon = (1-s)\sigma^2_\xi,$$

where the distorted belief $\hat{p}(\sigma_\epsilon^2)$ is such that

$$\hat{p}(\tau_\epsilon) \propto \exp\left(\lambda s^2 \sigma_\epsilon^2\right) p(\sigma_\epsilon^2).$$

Notice that, relative to the agents' prior $p(\sigma_\epsilon^2)$, the distorted belief $\hat{p}(\sigma_\epsilon^2)$ assigns higher weights to larger σ_ϵ^2 in Γ_ϵ: $\hat{p}(\sigma_\epsilon^2)$ first-order stochastically dominates $p(\sigma_\epsilon^2)$. It follows that

$$\int_{\Gamma_\epsilon} \sigma_\epsilon^2 \, \hat{p}(\sigma_\epsilon^2) \mathrm{d}\sigma_\epsilon^2 \geq \int_{\Gamma_\epsilon} \sigma_\epsilon^2 \, p(\sigma_\epsilon^2) \mathrm{d}\sigma_\epsilon^2 = \sigma_{\epsilon,*}^2 \ ,$$

and, therefore,

$$s^* = \frac{\sigma_\xi^2}{\sigma_\xi^2 + \int_{\Gamma_\epsilon} \sigma_\epsilon^2 \, \hat{p}(\sigma_\epsilon^2) \mathrm{d}\sigma_\epsilon^2} < \frac{\sigma_\xi^2}{\sigma_\xi^2 + \sigma_{\epsilon,*}^2} = s^{\mathrm{RE}}.$$

□

H Evidence on Inflation Expectations by Income Group

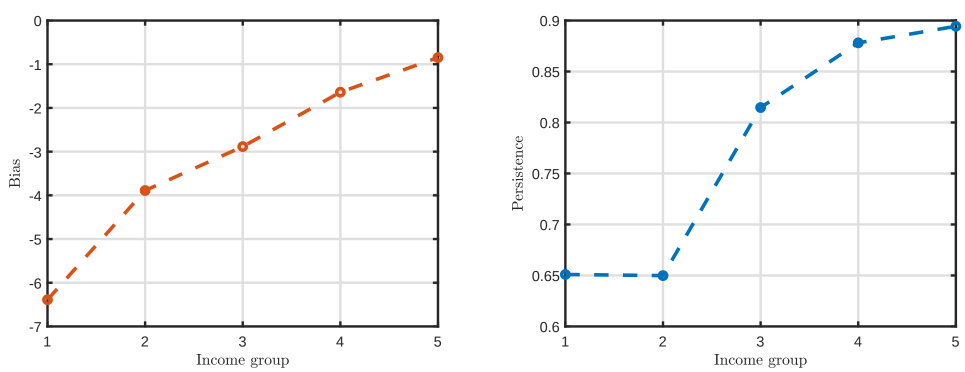
H.1 Forecast error bias and persistence

We investigate the joint behaviors of bias and persistence in forecast errors using both the Michigan Survey of Consumers (MSC) and the Survey of Consumer Expectations (SCE). We examine two regression equations:

$$\overline{\mathrm{FE}}_{g,t} = \sum_{g=1}^{N} \beta_g \mathcal{I}_g + \omega_{g,t},$$
$$\overline{\mathrm{FE}}_{g,t} = \sum_{g=1}^{N} \beta_g \mathcal{I}_g + \sum_{g=1}^{N} \alpha_g \overline{\mathrm{FE}}_{g,t-1} + \omega_{g,t},$$

where $\overline{\mathrm{FE}}_{g,t}$ represents the average forecast errors for group g at year-quarter t and $\mathcal{I}_g$ is the group dummy. For the MSC dataset, we divide individuals into $N = 7$ income groups, while for the SCE dataset, we divide individuals into $N = 5$ income groups. Table H.1 provides the results of our analysis. We use the poorest group (Group 1) as the reference group when reporting the results. The overall patterns of bias and persistence are similar in both the MSC and SCE datasets: as the income level increases, the amount of bias decreases, while the persistence of forecast errors increases. Similar to Figure 4.1 that displays the empirical patterns in MSC, Figure H.1 plot the point estimates of the biases and the persistence across different income groups in SCE.

FIGURE H.1: Bias and Persistence of Forecast Error in the Survey Data (NYSCE)



(A) Bias in forecast (B) Persistence of forecast error

Note: This figure reports bias (Panel A) and persistence (Panel B) of households' inflation forecasts in the cross-section of the income distribution. Bias and persistence of each income percentile are calculated by the mean and serial correlation of forecast errors of households' inflation expectations for the next 12 months. Data are obtained from the Survey of Consumer Expectations, NY Fed between 2013:II and 2022:II.

To address the concern that bias may be influenced by other observed individual characteristics, such as age and resident state, we introduce the following empirical specification at the individual level for both the MSC

Table H.1: Bias and Persistence of Forecast Errors: MSC and SCE

	MSC		SCE	
	Bias	Persistence	Bias	Persistence
Constant	-2.297^{***}	-1.055^{***}	-6.389^{***}	-2.250^{***}
	(0.072)	(0.0.081)	(0.162)	(0.208)
Group 2	0.235^{***}	0.367^{**}	2.500^{***}	0.865^{**}
	(0.060)	(0.085)	(0.108)	(0.153)
Group 3	0.766^{***}	0.565^{***}	3.504^{***}	1.722^{***}
	(0.053)	(0.053)	(0.143)	(0.140)
Group 4	1.103^{***}	0.713^{***}	4.750^{***}	2.090^{***}
	(0.057)	(0.032)	(0.140)	(0.063)
Group 5	1.258^{***}	0.810^{***}	5.541^{***}	2.194^{***}
	(0.054)	(0.025)	(0.163)	(0.109)
Group 6	1.535^{***}	0.876^{***}		
	(0.051)	(0.030)		
Group 7	1.924^{***}	0.959^{***}		
	(0.044)	(0.055)		
FE_{t-1}		0.537^{***}		0.651^{***}
		(0.044)		(0.031)
$FE_{t-1}\times$ Group 2		0.126^{***}		-0.001
		(0.041)		(0.068)
$FE_{t-1}\times$ Group 3		0.143^{***}		0.164
		(0.039)		(0.082)
$FE_{t-1}\times$ Group 4		0.171^{***}		0.227^{***}
		(0.023)		(0.050)
$FE_{t-1}\times$ Group 5		0.217^{***}		0.243^{**}
		(0.040)		(0.054)
$FE_{t-1}\times$ Group 6		0.219^{***}		
		(0.041)		
$FE_{t-1}\times$ Group 7		0.192^{***}		
		(0.042)		
Obs.	952	945	180	175

* p<0.1, ** p<0.05, *** p<0.001.

and the SCE:

$$\mathrm{FE}_{i,t} = \sum_{g=1}^{N} \beta_g \mathcal{I}_{i,g,t} + \gamma' \mathrm{X}_{i,t} + \delta_t + \omega_{i,t},$$

where $\mathcal{I}_{i,g,t}$ is a dummy variable that equals to 1 if individual i belongs to income group g at year-month t, and $\mathrm{X}_{i,t}$ is a vector of observed individual characteristics. For the MSC dataset, we control for age, gender, education, birth cohort, marital status, region, and the number of kids and adults in the household. It is worth noting that controlling for the birth cohort helps address concerns regarding the impact of inflation experiences on households' inflation expectations (Malmendier and Nagel, 2016). For the SCE dataset, we control for age group, numeracy, education, and region. Table H.2 reports the results. Again, we use the poorest group (Group 1) as the base group for both the MSC and SCE datasets. Even after controlling for additional individual characteristics, the biases in forecasts persist and exhibit a negative correlation with households' income levels.

Table H.2: Bias of Forecast Errors Controlling Individual Characteristics: MSC and SCE

	MSC	SCE
Constant	−2.370***	−5.222***
	(0.288)	(0.239)
Group 2	0.162***	1.819***
	(0.036)	(0.110)
Group 3	0.573***	2.421***
	(0.030)	(0.124)
Group 4	0.856***	3.200***
	(0.032)	(0.184)
Group 5	0.989***	3.728***
	(0.034)	(0.244)
Group 6	1.223***	
	(0.024)	
Group 7	1.510***	
	(0.031)	
Demographics	Yes	Yes
Birth Cohort	Yes	No
Age	Yes	Yes
Region	Yes	Yes
Time fixed effects	Yes	Yes
Obs.	146,622	134,190

* p<0.1, ** p<0.05, *** p<0.001.

H.2 CG and BGMS regressions

As a comparison to the group-specific CG and BGMS coefficients derived from our model, we run the corresponding CG and BGMS regressions using data from the Michigan Survey of Consumers and the Survey of Consumer Expectations. The term structure of the forecasts is not available in these datasets, preventing us from constructing exact forecast revisions. As a compromise, we consider the following closely related regressions instead:

$$\text{CG:} \quad \pi_{t+1} - \bar{\mathbb{E}}_t [\pi_{t+1}] = \alpha + \beta_{\text{CG}} \left(\bar{\mathbb{E}}_t [\pi_{t+1}] - \bar{\mathbb{E}}_{t-1} [\pi_t] \right) + \epsilon_{t+1}, \tag{H.1}$$

$$\text{BGMS:} \quad \pi_{t+1} - \mathbb{E}_{it} [\pi_{t+1}] = \alpha + \beta_{\text{BGMS}} \left(\mathbb{E}_{it} [\pi_{t+1}] - \mathbb{E}_{it-1} [\pi_t] \right) + \epsilon_{it+1}. \tag{H.2}$$

Columns (1)-(2) in Table H.3 display the results for the MSC, and columns (5)-(6) display the results for the SCE. At the individual level, the BGMS regression coefficients are more negative for poorer households, while the CG regression coefficients are larger for richer households. These results are broadly consistent with our model's predictions.

Table H.3: CG and BGMS Estimates: MSC and SCE

	MSC				SCE			
	(1) BGMS	(2) CG	(3) CG (IV)	(4) F-Stat	(5) BGMS	(6) CG	(7) CG (IV)	(8) F-Stat
Group 1	−0.555***	−0.411***	0.599*	14.08	−0.510***	−0.366***	−0.294	5.98
	(0.048)	(0.101)	(0.349)		(0.184)	(0.129)	(0.817)	
Group 2	−0.433***	−0.314**	2.030**	5.13	−0.438***	−0.267*	0.814	2.67
	(0.040)	(0.145)	(0.919)		(0.014)	(0.136)	(0.915)	
Group 3	−0.394***	−0.206	1.079**	9.04	−0.421***	−0.287	3.878	0.41
	(0.026)	(0.266)	(0.488)		(0.015)	(0.183)	(5.987)	
Group 4	−0.392***	−0.169	0.491*	24.87	−0.407***	0.299	3.030	3.61
	(0.031)	(0.230)	(0.285)		(0.019)	(0.330)	(2.920)	
Group 5	−0.378***	−0.147	0.982***	10.67	−0.376***	0.304	3.191*	12.44
	(0.028)	(0.260)	(0.382)		(0.035)	(0.341)	(1.900)	
Group 6	−0.399***	0.054	0.795**	17.56				
	(0.018)	(0.370)	(0.339)					
Group 7	−0.418***	0.011	0.980**	9.85				
	(0.017)	(0.301)	(0.481)					

* p<0.1, ** p<0.05, *** p<0.001.

However, due to the previously mentioned data limitations, the approximating regressions (H.1) and (H.2) may suffer from an endogeneity issue.[38] We follow Coibion and Gorodnichenko (2015) and use oil prices as the instrumental variable. Unfortunately, while the instrumental variable is strong enough for the entire sample,

[38]The error term ϵ_{t+1} in the CG specification above contains not only the rational expectations forecast errors $\hat{\epsilon}_{t+1}$ but also the expected change in inflation $\beta_{\text{CG}} \left(\bar{\mathbb{E}}_{t-1} [\pi_{t+1}] - \bar{\mathbb{E}}_{t-1} [\pi_t] \right)$. Under rational expectations, $\hat{\epsilon}_{t+1}$ is uncorrelated with the consensus forecast error $\pi_{t+1} - \bar{\mathbb{E}}_t [\pi_{t+1}]$. However, the covariance between the expected change in inflation $\beta_{\text{CG}} \left(\bar{\mathbb{E}}_{t-1} [\pi_{t+1}] - \bar{\mathbb{E}}_{t-1} [\pi_t] \right)$ and the consensus forecast error $\pi_{t+1} - \bar{\mathbb{E}}_t [\pi_{t+1}]$ is correlated as long as the inflation process is not a random walk. Therefore, the error term ϵ_{t+1} will be correlated with the forecast error on the left-hand side. Note that the reason for this endogeneity issue arises from the fact that neither the MSC nor the SCE provides the term structure of forecasts. As a result, forecasts are imperfectly overlapped.

it tends to be weak when segmenting the sample by different income groups (see the F-statistics in Columns (4) and (8)). Bearing in mind the weak IV issue, the CG coefficient generally increases with income, a trend that is more pronounced in the SCE data.

H.3 Balance-sheet effects

This section addresses the concern that balance-sheet effects may overturn the effects of inflation on labor income. We argue that this is unlikely to be the case.

First, notice that balance-sheet effects are primarily relevant for capital income, which constitutes a relatively small share of total income, especially for the income-poor. In Table H.4, we document the shares of different sources of income using the Survey of Consumer Finances, since this data is not available in the Michigan Survey.[39] For all households, capital and business income represent a relatively small share of total income, and this is especially true for the bottom four quintiles of income. The table also shows that the bottom quintiles of income have relatively low levels of net worth. With this in mind, one would expect that even the large proportional effects documented by Doepke and Schneider (2006) would be dominated by the effects of inflation on labor and transfer incomes.

TABLE H.4: Income Sources (%) by Quintiles of Income

	Quintiles of Income				
	1st	2nd	3rd	4th	5th
Labor	48.9	77.3	83.4	85.8	64.3
Capital	0.1	0.4	0.3	0.8	10.8
Business	6.2	5.4	5.9	5.6	18.7
Transfer	37.3	15.0	9.2	7.1	2.4
Other	7.5	1.8	1.2	0.7	3.7
Total Income	2.7	6.3	10.7	17.2	63.2
Net Worth	1.4	2.7	5.4	9.9	80.7

Notes: Calculated using data from the 2016 Survey of Consumer Finances. We use the definitions from Kuhn and Ríos-Rull (2016) and limit the sample to heads of households aged 18 to 65, for comparability with the results in the paper. We also choose the 2016 wave of the survey as it is roughly in the middle of the time sample we use in the paper.

One way to assess the overall effect of inflation on different households is to estimate a quantitative structural model incorporating the relevant mechanisms and heterogeneity, and then compute the conditional welfare effects for different groups. This approach is pursued by Cao, Meh, Ríos-Rull, and Terajima (2021). They find that poorer households are more negatively affected by inflation:

[39] The seven groups from the MSC sample have average incomes, in thousands of 2016 dollars, of {12.9, 24.5, 40.5, 59.8, 74.5, 104.9, 216.8}, while the quintiles of income from the SCF show averages of {13.9, 33.0, 56.4, 90.3, 331.2}. Although the top income levels from the SCF are higher, reflecting its detailed approach to top-coding issues, the bottom four quintiles align relatively well with the MSC groups. The net worth levels for these income quintiles in the SCF, again in thousands of 2016 dollars, are {39.2, 75.0, 156.2, 287.0, 2322.3}.

Table H.5: Concern About Inflation (%) by Income Levels

	Income Levels (thousands of dollars)							
	<25	25–35	35–50	50–75	75–100	100–150	150–200	>200
Very concerned	66.5	62.2	60.9	57.3	53.9	46.1	38.7	24.0
Somewhat concerned	17.9	20.6	21.7	23.0	21.9	26.2	27.3	28.2
A little concerned	10.4	12.6	13.1	14.4	16.7	19.1	22.2	29.5
Not at all concerned	5.2	4.5	4.2	5.3	7.5	8.6	11.8	18.3

Notes: Calculated using data from the 2024 House Pulse Survey. This survey started in 2020, so we selected the most recent wave to try to mitigate the impact of Covid-related concerns. Similar results are reported, using data from 2021, by Jayashankar and Murphy (2023).

> *"An increase in inflation from 2% to 5% costs 13% of one-year consumption. [...] From the point of view of consumption class, the poor lose a lot more than the rich: 37.0% of 2010 consumption versus 5.6% for the poorest and richest quintiles."*

The same conclusions can be drawn from the Census Bureau's Household Pulse Survey data, which includes the question, "In the area where you live and shop, how concerned are you, if at all, that prices will increase in the next six months?" In Table H.5, we present the results categorized by income brackets. A clear pattern can be observed, with the inflation concern monotonically decreasing as income levels increase.